On constructions and parameters of symmetric configurations v_k

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Abstract: The spectrum of possible parameters of symmetric configurations is investigated. We both survey known constructions and results, and propose some new construction methods. Many new parameters are obtained, in particular for cyclic symmetric configurations, which are equivalent to deficient cyclic difference sets. Both Golomb rulers and modular Golomb rulers are a key tool in our investigation. Several new upper bounds on the minimum integer $E(k)$ such that for each $v \geq E(k)$ there exists a symmetric configuration v_k are obtained. Upper bounds of the same type are provided for cyclic symmetric configurations. From the standpoint of applications, it should be noted that our results extend the range of possible parameters of LDPC codes, generalized LDPC codes, and quasi-cyclic LDPC codes.

Keywords: *configurations in Combinatorics; symmetric configurations; cyclic configurations; Golomb rulers; modular Golomb rulers*

1 Introduction

Configurations are interesting combinatorial structures. They were defined in 1876. For an introduction to the problems connected with the configurations and their history, see [28, 29, 30] and the references therein.

Definition 1.1. [29]

(i) A configuration (v_r, b_k) is an incidence structure of v points and b lines such that each line contains k points, each point lies on r lines, and two distinct points are connected by *at most* one line.

(ii) If $v = b$ and, hence, $r = k$, the configuration is *symmetric*, and it is referred to as a configuration v_k.

(iii) The *deficiency* d of a configuration (v_r, b_k) is the value $d = v - r(k-1) - 1$.

A symmetric configuration v_k is *cyclic* if there exists a permutation of the set of its points mappings blocks to blocks, and acting regularly on both points and blocks. Equivalently, v_k is cyclic if one of its incidence matrix is circulant.

Steiner systems are configurations with $d = 0$ [29]. The deficiency of a symmetric configuration v_k is $d = v - (k^2 - k + 1)$. The deficiency of v_k is zero if and only if v_k is a finite projective plane of order $k - 1$. In general, d indicates the number of points not joined with an arbitrary point or the number of lines parallel to an arbitrary line, see [2, 19, 24, 29].

A configuration (v_r, b_k) can be treated also as a k-uniform r -regular linear hypergraph with v vertices and b hyperedges [27, 29]. Connections of configurations (v_r, b_k) with numerical semigroups are noted in [10]. Some analogies between configurations (v_r, b_k), regular graphs, and molecule models of chemical elements are remarked in [25]. As an example of a practical applying configurations (symmetric and nonsymmetric) we mention also the problem of user privacy for using database, see [15, 50] and the references therein.

Denote by $\mathbf{M}(v, k)$ an incidence matrix of a symmetric configuration v_k. Any matrix $\mathbf{M}(v, k)$ is a $v \times v$ 01-matrix with k units in every row and column; moreover, the 2×2 matrix $\mathbf{J}_2$ consisting of all units is not a submatrix of $\mathbf{M}(v, k)$. Therefore, $\mathbf{M}(v, k)$ is a $\mathbf{J}_2$-*free matrix.* Two incidence matrices of the same configuration may differ by a permutation on the rows and the columns.

A matrix $\mathbf{M}(v, k)$ can also be considered as a biadjacency matrix of a k-regular bipartite graph without multiple edges. The biadjacency matrix describes connections of two vertex subsets of the graph so that the adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0}_v & \mathbf{M}(v,k) \\ \mathbf{M}^{tr}(v,k) & \mathbf{0}_v \end{bmatrix}$$

where tr stands for transposition, and $\mathbf{0}_v$ denotes the zero $v \times v$ matrix. This graph is the *Levi graph* of the configuration v_k [29, Sec. 7.2]. As $\mathbf{M}(v, k)$ is a $\mathbf{J}_2$-free, the graph has girth at least six, i.e. it does not contain 4-cycles. Such graphs are useful for the construction of bipartite-graph codes that can be treated as *low-density parity-check* (LDPC) *codes* or *generalized* LDPC codes [3]–[6],[12, 14, 20, 35, 41]. If $\mathbf{M}(v, k)$ is *circulant*, then the corresponding LDPC code is *quasi-cyclic*; it can be encoded with the help of shift-registers with relatively small complexity, see [12, 14, 20, 35] and the references therein.

Matrices $\mathbf{M}(v, k)$ consisting of square circulant submatrices have a number of useful properties, e.g. they are more suitable for LDPC codes implementation. We say that a 01-matrix $\mathbf{A}$ is *block double-circulant* (BDC for short) if $\mathbf{A}$ consists of square circulant blocks whose weights give rise to a circulant matrix (see Definition 3.1 for details). A configuration v_k with a BDC incidence matrix $\mathbf{M}(v, k)$ is called a *BDC symmetric configuration.* Symmetric and non-symmetric configurations with incidence matrices consisting of square circulant blocks are considered, e.g. in [12]–[14],[43].

Cyclic configurations are considered, for instance, in [12]–[14],[18, 24, 38, 40]. A standard method to construct cyclic configurations (or, equivalently, circulant matrices $M_{v,k}$) is based on *Golomb rulers* [16, 18, 22, 24],[45]–[47].

Definition 1.2. [45, 18]

(i) A *Golomb ruler* G_k of *order* k is an ordered set of k integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \mid 1 \leq j < i \leq k\}$ are distinct. The *length* $L_{\mathrm{G}}(k)$ of the ruler G_k is equal to $a_k - a_1$.

(ii) A Golomb ruler G_k is an *optimal Golomb ruler* OG_k if no shorter Golomb ruler of the same order k exists. Let $L_{\mathrm{OG}}(k)$ and $L_{\overline{\mathrm{G}}}(k)$ be the length of an optimal ruler OG_k and of the *shortest known* Golomb ruler $\overline{\mathrm{G}}_k$, respectively.

(iii) A (v, k) *modular Golomb ruler* is an ordered set of k integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \mid 1 \leq i, j \leq k;\ i \neq j\}$ are distinct and nonzero modulo v.

Clearly, $L_{\overline{\mathrm{G}}}(k) \geq L_{\mathrm{OG}}(k)$ holds.

For any value $\delta \geq 0$, Golomb rulers $(a_1, a_2, \ldots, a_k)$ and $(a_1 + \delta, a_2 + \delta, \ldots, a_k + \delta)$ have the same properties. Usually, $a_1 = 0$ is assumed.

Remark 1.3. A (v, k) modular Golomb ruler is also called a *deficient cyclic difference set* with deficiency $d = v - (k^2 - k + 1)$. For a deficient cyclic difference set the deficiency d is the number of elements in $\mathbb{Z}_v \setminus \{0\}$ not represented by any difference $a_i - a_j$ [18]. Note that the expression "deficient cyclic difference set" is used in [18], whereas in [38] and [40] the expressions "difference set modulo v " and "deficient difference set in $\mathbb{Z}_v$" are adopted.

Remark 1.4. Golomb rulers and modular Golomb rulers are deeply connected with difference triangle sets and difference packings, see e.g. [37, 45, 48]. In particular, according to the notation of [45], a Golomb ruler G_k is a difference triangle set $(1, k-1)$-DΔS, whereas a (v, k) modular Golomb ruler is a difference packing 1-DP(v, k) [45, Prop. 19.9, Rem. 19.24]. If $a_1 = 0$ the corresponding object is said to be *normalized.* Note also that in [51], the expression "planar cyclic difference packing modulo v" is used for an object equivalent to a (v, k) modular Golomb ruler.

In [16] it is proved that

$$L_{\mathrm{OG}}(k) > k^2 - 2k\sqrt{k} + \sqrt{k} - 2.$$

Currently, the optimal lengths $L_{\mathrm{OG}}(k)$ are known only for orders $k \leq 25$ [16, 22, 45, 46]. So, for $k \leq 25$ we have $L_{\overline{\mathrm{G}}}(k) = L_{\mathrm{OG}}(k)$. The proof of the optimality of a Golomb ruler is a hard problem needing exhaustive computer search. On the other hand, for sufficiently large orders k, relatively short Golomb rulers are constructed and are available online, see e.g. the internet resources [16, 22, 46, 47] and the references therein. For $k \leq 150$, the

order of magnitude of the lengths $L_{\overline{\mathrm{G}}}(k)$ of the shortest known Golomb rulers is ck^2 with $c \in [0.7, 0.9]$, see [16, 18, 22, 24, 29, 45, 46]. Moreover,

$$L_{\mathrm{OG}}(k) \le L_{\overline{\mathrm{G}}}(k) < k^2 \text{ for } k < 65000,$$

see [16]. Other constructions for large k can be found in [17].

We say that a 0,1-*vector* $\mathbf{u} = (u_0, u_1, \dots, u_{v-1})$ *corresponds* to a (modular) Golomb ruler if the increasing sequence of integers $j \in \{0, 1, \dots, v-1\}$ such that $u_j = 1$ form a (modular) Golomb ruler.

Recall that *weight* of a *circulant* $0,1$-*matrix* is the number of units in each its row.

Theorem 1.5. [24, Sec. 4]

(i) *Any Golomb ruler* G_k *of length* $L_{\mathrm{G}}(k)$ *is a* (v, k) *modular Golomb ruler for all* v *such that* $v \ge 2L_{\mathrm{G}}(k) + 1$.

(ii) *A circulant* $v \times v$ 0,1-*matrix of weight* k *is an incidence matrix* $\mathbf{M}(v, k)$ *of a cyclic symmetric configuration* v_k *if and only if the first row of the matrix corresponds to a* (v, k) *modular Golomb ruler.*

We remark that (ii) of Theorem 1.5 is not explicitly stated in [24]. However, the assertion can be easily deduced from the results in [24].

Corollary 1.6. [24, Sec. 4] *For all* v *such that*

$$v \ge 2L_{\overline{\mathrm{G}}}(k) + 1, \tag{1.1}$$

there exists a cyclic symmetric configuration v_k.

We call the value $G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1$ the *Golomb bound*.

It is well known that $v \ge k^2 - k + 1$ holds for configurations v_k, and that the lower bound is attained if and only if there exists a projective plane of the order $k - 1$ [24, 29]. We call $P(k) = k^2 - k + 1$ the *projective plane bound*.

Let $v_\delta(k)$ be the smallest possible value of v for which a (v, k) modular Golomb ruler (or, equivalently, a cyclic symmetric configuration) exists.

Finally, we introduce other two bounds. The *existence bound* $E(k)$ is the integer such that for any $v \ge E(k)$, there exists a symmetric configuration v_k. Similarly, the *cyclic existence bound* $E_c(k)$ is the integer such that for any $v \ge E_c(k)$, there exists a cyclic v_k.

Clearly, for a fixed k, we have

$$k^2 - k + 1 = P(k) \le E(k) \le E_c(k) \le G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1. \tag{1.2}$$

$$k^2 - k + 1 = P(k) \le v_\delta(k) \le E_c(k) \le G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1. \tag{1.3}$$

The aim of this work is threefold:

- to survey the vast body of literature on constructions and parameters of symmetric configurations v_k;

- to describe new construction methods, paying special attention to constructions producing circulant and block double-circulant incidence matrices $\mathbf{M}(v,k)$;

- to investigate the *spectrum* of possible parameters of symmetric configurations v_k (with special attention to parameters of cyclic symmetric configurations) in the interval
$$k^2 - k + 1 = P(k) \leq v < G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1. \tag{1.4}$$

Our main achievements are new constructions of BDC incidence matrices (see Theorems 3.5 and 3.11, together the examples in Section 3), improvements on the known upper bounds on $E(k)$ and $E_c(k)$, and several new parameters for cyclic and non-cyclic configurations v_k, see Sections 5 and 6.

From the stand point of applications, including Coding Theory, it is sometimes useful to have different matrices $\mathbb{M}(v,k)$ for the same v and k. This is why we attentively consider various constructions, even when they provide configurations with the same parameters.

The *Extension Construction*, as introduced in [3, 4], plays a key role for investigation of the spectrum of possible parameters of symmetric configurations v_k, $k \geq 11$, as it provides *intervals* of values of v for a fixed k. To be successfully applied, the Extension Construction needs a convenient starting incidence matrix. Block double-circulant matrices turn out to be particularly useful in this context, see Corollary 4.3. In this work we use both the original starting matrices of [3, 4] and some new ones obtained by our new constructions, see Example 4.4.

We remark that new cyclic configurations provide new modular Golomb rulers, i.e. new deficient cyclic difference sets. Note also that methods considered in this work could be also used to construct non-symmetric configurations (v_r, b_k).

The paper is organized as follows. In Section 2, the known constructions and parameters of configurations v_k are considered. In particular, a geometrical construction from [12, 14], and the Extension Construction from [3, 4] are described. In Section 3, some new constructions of block double-circulant incidence matrices $\mathbf{M}(v,k)$ are proposed. In Section 4, methods for constructing matrices admitting extensions are described. In Sections 5 and 6, our results on the spectra of parameters of cyclic and non-cyclic configurations are reported. An Appendix contains the proof of one of theorems from Section 3.

Some results of this work were published without proofs in [12, 13].

2 Some known constructions and parameters of configurations v_k with $P(k) \leq v < G(k)$

The aim of this section is to provide a list of pairs (v,k) for which a (cyclic) symmetric configuration v_k is known to exist, see Equations (2.1)-(2.13). In most cases a brief description of the corresponding configuration is given. Infinite families of configuration v_k given in this section are considered in [1]–[4],[8, 12, 21, 38, 40],[14]–[19],[24]–[29]; see also the references therein.

Throughout the paper, q is a prime power and p is a prime. Let F_q be Galois field of q elements. Let $F_q^* = F_q \setminus \{0\}$. Let $\mathbf{0}_u$ be the zero $u \times u$ matrix. Denote by $\mathbf{P}_u$ a permutation matrix of order u.

First we recall that several pairs $(v, k-\delta)$ can be actually obtained from a given v_k.

Theorem 2.1. [40] *If a (cyclic) configuration v_k exists, then for each δ with $0 \leq \delta < k$ there exists a family of (cyclic) configurations $v_{k-\delta}$ as well.*

At once we note that a cyclic configuration v_k gives a family of cyclic configurations $v_{k-\delta}$ obtained by distmissng δ units in the 1-st row of its incidence matrix. For the general case, Theorem 2.1 is based on the fact that an incidence matrix $\mathbf{M}(v,k)$ can be represented as a sum of k permutations $v \times v$ matrices (in many ways). This fact follows from the results of Steinits (1894) and König (1914), see e.g. [27, Sec. 5.2] and [30, Sec. 2.5].

The value δ appearing in Equations (2.1)-(2.13) is connected with Theorem 2.1. When a reference is given, it usually refer to the case $\delta = 0$.

The families giving rise to pairs (2.1)–(2.3) below are obtained from (v, k) modular Golomb rulers [16, Ch. 5],[17],[24, Sec. 5],[45, Sec. 19.3], see Theorem 1.5(ii).

$$\text{cyclic } v_k \;:\; v = q^2 + q + 1, \;\; k = q + 1 - \delta, \;\; q + 1 > \delta \geq 0; \tag{2.1}$$

$$\text{cyclic } v_k \;:\; v = q^2 - 1, \;\; k = q - \delta, \;\; q > \delta \geq 0; \tag{2.2}$$

$$\text{cyclic } v_k \;:\; v = p^2 - p, \;\; k = p - 1 - \delta, \;\; p - 1 > \delta \geq 0. \tag{2.3}$$

The configurations giving rise to (2.1) use the incidence matrix of the cyclic projective plane $PG(2,q)$ [49],[16, Sec. 5.5],[17],[45, Th. 19.15]. The family with parameters (2.2) is obtained from the *cyclic starred affine plane* $AG(2,q)$ [8],[16, Sec. 5.6],[17, 18],[45, Th. 19.17], see also [14, Ex. 5] and [19] where the configurations are called *anti-flags*. We recall that the starred plane $AG(2,q)$ is the affine plane without the origin and the lines through the origin. Finally, the configurations with parameters (2.3) follow from Ruzsa's construction [44],[16, Sec. 5.4],[17],[45, Th. 19.19].

The non-cyclic families with parameters (2.4) and (2.5) are given in [1, Constructions (i),(ii), p. 126] and [21, Constructions 3.2,3.3, Rem. 3.5], see also the references therein and [3],[4, Sec. 3],[14, Sec. 7.3].

$$v_k \;:\; v = q^2 - qs, \;\; k = q - s - \delta, \;\; q > s \geq 0, \;\; q - s > \delta \geq 0; \tag{2.4}$$

$$v_k \;:\; v = q^2 - (q-1)s - 1, \;\; k = q - s - \delta, \;\; q > s \geq 0, \;\; q - s > \delta \geq 0. \tag{2.5}$$

In the projective plane $PG(2,q)$ we fix a line ℓ and a point P and assign an integer $s \geq 0$. If $P \in \ell$ we choose s points on ℓ distinct from P, and s lines through P distinct from ℓ. If $P \notin \ell$ we choose s arbitrary points on ℓ and consider the s lines connecting P with these points. The incidence structure obtained from $PG(2,q)$ by dismissing all the lines through the $s+1$ selected points and all the points lying on the $s+1$ selected lines provides the family of (2.4) if $P \in \ell$ and the family with parameters (2.5) if $P \notin \ell$. For $s = 0$, the construction of (2.5) is given in [40]. In [3],[4, Eqs (3.2),(3.3)], the family with parameters

(2.4) is described by using a block structure of the incidence matrix of the affine plane $AG(2,q)$, see the Extension Construction below. Configurations $(q^2)_q$ and $(q^2-1)_q$ are mentioned in many papers, see e.g. [19],[24, Sec. 5].

For q a square, in [1, Conjec. 4.4, Rem. 4.5, Ex. 4.6] and [21, Construction 3.7, Th. 3.8], families of non-cyclic configuration v_k with parameters (2.6) are provided; see also [14, Ex. 8]. The configurations with parameters (2.7) belong to these families; here, $c = q-\sqrt{q}$. Configurations with parameters (2.7) are also described in [12, Ex. 2(ii)] and [19].

$$v_k \ : \ v = c(q+\sqrt{q}+1), k = \sqrt{q}+c-\delta, c = 2,3,\dots,q-\sqrt{q}, \delta \geq 0, q \text{ square}; \quad (2.6)$$
$$v_k \ : \ v = q^2-\sqrt{q}, \ \ k = q-\delta, \ \ q > \delta \geq 0, \ \ q \text{ square}. \quad (2.7)$$

In [1, 12, 14, 19, 21], the partition $PG(2,q)$ into Baer subplanes for q a square is used; see also Example 3.9(ii) of the present work.

In [19, Th. 1.1], a family of non-cyclic with parameters

$$v_k : v = 2p^2, \ \ k = p+s-\delta, \ \ 0 < s \leq q+1, \ \ q^2+q+1 \leq p, \ \ p+s > \delta \geq 0 \quad (2.8)$$

is given. In [14, Sec. 6], based on the cyclic starred affine plane, a construction of non-cyclic configuration with parameters

$$\begin{aligned} v_k \ &: \ v = c(q-1), \ \ k = c-\delta, \ \ c = 2,3,\dots,b, \ \ b = q \text{ if } \delta \geq 1, \quad (2.9)\\ b \ &= \ \left\lceil \frac{q}{2} \right\rceil \text{ if } \delta = 0, \ \ c > \delta \geq 0, \end{aligned}$$

is provided.

In [12, Sec. 2],[14, Sec. 3], the following geometrical construction which uses point orbits under the action of a collineation group is described.

Construction A. Take any point orbit $\mathcal{P}$ under the action of a collineation group in an affine or projective space of order q. Choose an integer $k \leq q+1$ such that the set $\mathcal{L}(\mathcal{P},k)$ of lines meeting $\mathcal{P}$ in precisely k points is not empty. Define the following incidence structure: the points are the points of $\mathcal{P}$, the lines are the lines of $\mathcal{L}(\mathcal{P},k)$, the incidence is that of the ambient space.

Theorem 2.2. *In Construction A the number of lines of $\mathcal{L}(\mathcal{P},k)$ through a point of $\mathcal{P}$ is a constant r_k. The incidence structure is a configuration (v_{r_k}, b_k) with $v = |\mathcal{P}|$, $b = |\mathcal{L}(\mathcal{P},k)|$.*

By Definition 1.1, if $r_k = k$ Construction A produces a symmetric configuration v_k.

It is noted in [12, 14] that Construction A works for any 2-$(v,k,1)$ design D and for any group of automorphisms of D. The role of $q+1$ is played by the size of any block in D.

Families of non-cyclic configuration v_k obtained by Construction A with the following parameters are given in [14, Exs 2, 3].

$$v_k \ : \ v = \frac{q(q-1)}{2}, \ \ k = \frac{q+1}{2} - \delta, \ \ \frac{q+1}{2} > \delta \geq 0, \ \ q \text{ odd}. \quad (2.10)$$
$$v_k \ : \ v = \frac{q(q+1)}{2}, \ \ k = \frac{q-1}{2} - \delta, \ \ \frac{q-1}{2} > \delta \geq 0, \ \ q \text{ odd}. \quad (2.11)$$
$$v_k \ : \ v = q^2+q-q\sqrt{q}, \ \ k = q-\sqrt{q}, \ \ q-\sqrt{q} > \delta \geq 0, \ \ q \text{ square}. \quad (2.12)$$

In [3], a construction method for non-cyclic configuration v_k with parameters (2.13) is proposed, and called "Construction θ-extension". This construction is also considered in [4], where it is called CE-construction ("Cancellation+Enlargement"). The terminology we use here is "*Extension Construction*".

$$v_k : v = q^2-qs+\theta,\ k = q-s-\Delta,\ q > s \geq 0,\ q-s > \Delta \geq 0,\ \theta = 0, 1, \ldots, q-s+1. \quad (2.13)$$

We first describe the Extension Construction in geometrical terms. Let v_k be a configuration $(\mathcal{P}, \mathcal{L})$ with incidence matrix $\mathbf{M}(v, k)$. Assume that there exists a set of $k-1$ pairwise disjoint lines of v_k, say $\ell_1, \ell_2, \ldots, \ell_{k-1}$, and a set of $k-1$ pairwise non-collinear points, say $P_1, P_2, \ldots, P_{k-1}$, with the property that each P_i belongs to precisely one $\ell_{\pi(i)}$. Here π denotes a permutation of the indexes $1, 2, \ldots, k-1$. We define a new incidence structure $(\mathcal{P}', \mathcal{L}')$ as follows:

1. $\mathcal{P}' = \mathcal{P} \cup \{P_{\text{new}}\}$;
2. $\mathcal{L}' = \mathcal{L} \cup \{\ell_{\text{new}}\}$;
3. the lines incident with P_{new} are $\ell_1, \ldots, \ell_{k-1}$ and ℓ_{new};
4. the points incident with ℓ_{new} are $P_1, \ldots, P_{k-1}$ and P_{new};
5. P_i is not incident with $\ell_{\pi(i)}$;
6. for a point $P \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$ we have that P is incident with ℓ if and only if $P \in \ell$ in v_k, with the only $k-1$ exceptions of $P = P_i$ and $\ell = \ell_{\pi(i)}$, $i = 1, \ldots, k-1$.

It is easy to check that $(\mathcal{P}', \mathcal{L}')$ is a configuration $(v+1)_k$.

It is interesting to note that this procedure can be viewed as a generalization of a classical construction by V. Martinetti for configurations v_3, going back to 1887 [39]. According to Martinetti's construction (quoted, e.g. in [7, 9, 25],[30, Sec. 2.4, Fig. 2.4.1]) two parallel lines a, b and two non collinear points A, B are chosen so that $A \in a$, $B \in b$. Then a line c and a point C are added. The points A, B are removed from the lines a and b and are included into the new line c. The new point C is included into all lines a, b, and c.

Below we provide a description of the Extension Construction, as given in [3, 4].

Definition 2.3. [3, 4] Let $\mathbf{M}(v, k)$ be an incidence matrix of a symmetric configuration v_k. In $\mathbf{M}(v, k)$, we consider an aggregate $\mathcal{A}$ of $k-1$ rows corresponding to pairwise disjoint lines of v_k and $k-1$ columns corresponding to pairwise non-collinear points of v_k. The $(k-1) \times (k-1)$ submatrix $\mathbf{C}(\mathcal{A})$ formed by the intersection of the rows and columns of $\mathcal{A}$ is called a *critical submatrix* of $\mathcal{A}$. The aggregate $\mathcal{A}$ is called an *extending aggregate* (or *E-aggregate*) if its critical submatrix $\mathbf{C}(\mathcal{A})$ is a permutation matrix $\mathbf{P}_{k-1}$. The matrix $\mathbf{M}(v, k)$ *admits an extension* if it contains at least one E-aggregate. The matrix $\mathbf{M}(v, k)$ *admits θ extensions* if it contains θ E-aggregates that do not intersect each other. We also will say that a configuration v_k *admits an extension* or *admits θ extensions* if its incidence matrix does.

Procedure E (*Extension Procedure*). Let $\mathbf{M}(v,k)=[m_{ij}]$ be an incidence matrix of a symmetric configuration $v_k=(\mathcal{P},\mathcal{L})$. Assume that $\mathbf{M}(v,k)$ admits an extension.

1. To the matrix $\mathbf{M}(v,k)$, add a new row from below and a new column to the right. Denote the new $(v+1)\times(v+1)$ matrix by $\mathbf{B}=[b_{ij}]$, and let $b_{v+1,v+1}=1$ whereas $b_{v+1,1}=\ldots=b_{v+1,v}=0$, $b_{1,v+1}=\ldots=b_{v,v+1}=0$.

2. One of E-aggregates of $\mathbf{M}(v,k)$, say $\mathcal{A}$, is chosen. In the matrix $\mathbf{B}$, we "clone" all $k-1$ units of the critical submatrix $\mathbf{C}(\mathcal{A})$ writing their "projections" to the new row and column. Finally, the units cloned are changed by zeroes. In other words, let the aggregate $\mathcal{A}$ consist of rows with indexes i_u, $u=1,2,\ldots,k-1$, and columns with indexes j_d, $d=1,2,\ldots,k-1$. Then the units of $\mathbf{C}(\mathcal{A})$ are as follows: $m_{i_u j_{\pi(u)}}=1$, $u=1,2,\ldots,k-1$, for some permutation π of the indexes $1,\ldots,k-1$. Then $\mathbf{B}$ arising from Step 1 is changed as follows: $b_{i_u,v+1}=1$, $b_{v+1,j_d}=1$, $b_{i_u j_{\pi(u)}}=0$, $u=1,2,\ldots,k-1$, $d=1,2,\ldots,k-1$.

It is easily seen that $\mathbf{B}$ is an incidence matrix for $(\mathcal{P}',\mathcal{L}')$. Therefore, the following result can be easily proved.

Theorem 2.4. [3, 4] *Let $\mathbf{M}(v,k)$ be an incidence matrix of a symmetric configuration v_k. Assume that $\mathbf{M}(v,k)$ admits θ extensions, for some $\theta\geq 1$.*

(i) *θ repeated applications of Procedure E to $\mathbf{M}(v,k)$ gives an incidence matrix $\mathbf{M}(v+\theta,k)$ of a symmetric configuration $(v+\theta)_k$.*

(ii) *If $\theta\geq k-1$, then any $k-1$ new rows and $k-1$ new columns obtained as a result of repeated application of Procedure E form an E-aggregate.*

In [3, 4] the Extension Construction is applied to affine planes and provides configuration with parameters as in (2.13). Let (x_1,x_2) denote coordinates for the affine plane $AG(2,q)$. The incidence structure of $AG(2,q)$ is a resolvable 2-$(q^2,q,1)$-design with $q+1$ resolution classes. Each class contains q parallel lines. The q classes are lines with equation $x_2=wx_1+u$ where $w\in F_q$ is a constant for the given class and u runs over F_q. One more class contains q lines $x_1=c$. This class is removed from $AG(2,q)$ in order to obtain a symmetric configuration $(q^2)_q$, whose incidence matrix $\mathbf{M}(q^2,q)$ can be represented as a superposition of q^2 permutation matrices $\mathbf{P}_q$. Each block row contains one resolution class. Each block column corresponds to q points (d,x_2) where d is a constant for the given block column and x_2 runs over F_q. Then from $\mathbf{M}(q^2,q)$ one removes s block rows and columns. An incidence matrix $\mathbf{M}(q^2-qs,q-s)$ is obtained. It is a superposition of $(q-s)^2$ matrices $\mathbf{P}_q$. Further, a $(q-s)\times(q-s)$ 01-matrix $\mathbf{S}_\Delta$ with Δ units in every row and column is taken. In $\mathbf{M}(q^2-qs,q-s)$, submatrices $\mathbf{P}_q$ marked by units of $\mathbf{S}_\Delta$ are changed by $\mathbf{0}_q$. An incidence matrix $\mathbf{M}(q^2-qs,q-s-\Delta)$. is obtained; it admits $\theta\leq q-s$ extensions. When Procedure E is executed by $q-s$ times, Procedure E can be applied once more, according to Theorem 2.4(ii).

Some known results on existence and non-existence of sporadic symmetric configurations will be mentioned in Sections 5 and 6.

We end this section by remarking that cyclic symmetric configurations can be constructing from Sidon sets. Sidon sets are combinatorial objects equivalent to Golomb rulers.

Definition 2.5. [16, 42] A *Sidon k-set* (respectively, (v,k) *modular Sidon set*) is an ordered set of k integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all pairwise sums $\{a_i + a_j \mid 1 \leq i \leq j \leq k\}$ are different (respectively, different modulo v).

Sidon sets are called also *Sidon sequences*, or *B_2 sequence*; see [16, 42] and the references therein for more details and terminology. It should be noted that in Sidon sets we consider sums $a_i + a_j$ of not necessarily distinct elements.

The relation between Sidon sets and Golomb rulers is described in the following well-known result (for a proof see e.g. [16, Ch. 4]).

Theorem 2.6. *A Sidon k-set (respectively, (v,k) modular Sidon set) is a Golomb ruler of order k (respectively, (v,k) modular Golomb ruler), and conversely.*

The smallest possible value of v for which a (v,k) modular Sidon set exists coincides with $v_\delta(k)$. This makes our notation consistent with [23]. General bounds on $v_\delta(k)$ and precise results for smaller k's can be found in [23, 31, 47, 51].

3 Constructions of block double-circulant incidence matrices $\mathbf{M}(v,k)$

The aim of this section is the construction of BDC incidence matrices of cyclic symmetric configurations. A method based on the Golomb ruler associated to a cyclic simmetric configuration is described in Subsection 3.2: splitting a starting modular Golomb ruler a number of quotient Golomb rulers forming a needed BDC matrix are obtained. The same method can be described in terms of the action of the automorphism group of the configuration, see Subsection 3.3. We provide two different descriptions because when dealing with a given configuration usually either one approach or the other can be more conveniently used. For example, ideas of Subsection 3.2 work better if the Golomb ruler associated to a configuration is described explicitly as a list of integers. The approach of Subsection 3.3 is useful for instance when the configuration arises from geometrical objects such as cyclic projective and affine planes. Sometimes both the approaches can be conveniently used, cf. Examples 3.7 and 3.12(i).

Throughout this section,

$$(a_1, a_2, \ldots, a_k) \text{ is a } (v,k) \text{ modular Golomb ruler, with } v = td \text{ for integers } t, d. \quad (3.1)$$

3.1 BDC matrices $\mathbf{M}(v,k)$ and families of symmetric configurations

Definition 3.1. Let $v = td$. A $v \times v$ matrix $\mathbf{A}$ is said to be a *block double-circulant matrix* (or *BDC matrix*) if

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_{0,0} & \mathbf{C}_{0,1} & \dots & \mathbf{C}_{0,t-1} \\ \mathbf{C}_{1,0} & \mathbf{C}_{1,1} & \dots & \mathbf{C}_{1,t-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}_{t-1,0} & \mathbf{C}_{t-1,1} & \dots & \mathbf{C}_{t-1,t-1} \end{bmatrix}, \tag{3.2}$$

where $\mathbf{C}_{i,j}$ is a *circulant* $d \times d$ 0,1-matrix for all i, j, and submatrices $\mathbf{C}_{i,j}$ and $\mathbf{C}_{l,m}$ with $j - i \equiv m - l \pmod t$ have equal weights. The matrix

$$\mathbf{W}(\mathbf{A}) = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & \dots & w_{t-2} & w_{t-1} \\ w_{t-1} & w_0 & w_1 & w_2 & \dots & w_{t-3} & w_{t-2} \\ w_{t-2} & w_{t-1} & w_0 & w_1 & \dots & w_{t-4} & w_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1 & w_2 & w_3 & w_4 & \dots & w_{t-1} & w_0 \end{bmatrix} \tag{3.3}$$

is a *circulant* $t \times t$ matrix whose entry in position i, j is the *weight* of $\mathbf{C}_{i,j}$. $\mathbf{W}(\mathbf{A})$ is called the *weight matrix* of $\mathbf{A}$. The vector $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, w_1, \dots, w_{t-1})$ is called the *weight vector* of $\mathbf{A}$.

We present some simple techniques for obtaining BDC matrices of symmetric configurations from a given BDC $v \times v$ matrix $\mathbf{A}$ of (3.2) with weight matrix $\mathbf{W}(\mathbf{A})$ of (3.3). We assume that $v = td$.

(i) For $h \in \{0, 1, \dots, t-1\}$, dismiss $\delta_h \geq 0$ units in each row of every submatrix $\mathbf{C}_{i,j}$ with $j - i \equiv h \pmod t$, in such a way that the the obtained submatrix is still circulant. A BDC matrix $\mathbf{A}'$ is then obtained; it consists of circulant matrices $\mathbf{C}'_{i,j}$ of weight $w'_h = w_h - \delta_h$, where $j - i \equiv h \pmod t$ and $h = 0, 1, \dots, t-1$. It is an incidence BDC matrix of a configuration $v'_{k'}$ with $\overline{\mathbf{W}}(\mathbf{A}') = (w'_0, w'_1, \dots, w'_{t-1})$,

$$v' = v, \ k' = k - \sum_{h=0}^{t-1} \delta_h, \ 0 \leq \delta_h \leq w_h, \ w'_h = w_h - \delta_h. \tag{3.4}$$

(ii) Fix some non-negative integer $j \leq t-1$. Let m be such that $w_m \leq w_h$ for all $h \neq j$. Cyclically shift all block rows of $\mathbf{A}$ to the left by j block positions. A matrix $\mathbf{A}^*$ with $\overline{\mathbf{W}}(\mathbf{A}^*) = (w^*_0 = w_j, w^*_1 = w_{j+1}, \dots, w^*_u = w_{u+j \pmod t}, \dots, w^*_{t-1} = w_{j-1})$ is obtained. By applying (i), construct a matrix $\mathbf{A}^{**}$ with $w^{**}_0 = w^*_0 = w_j$, $w^{**}_h = w_m$, $h \geq 1$. Now remove from $\mathbf{A}^{**}$ $t - c$ block rows and columns from the bottom and the right. In this way a $cd \times cd$ BDC matrix $\mathbf{A}'$ is obtained, with $\overline{\mathbf{W}}(\mathbf{A}') = (w_j, w_m, \dots, w_m)$. It is an incidence matrix of a configuration $v'_{k'}$ with

$$v' = cd, \ k' = w_j + (c-1)w_m, \ c = 1, 2, \dots, t. \tag{3.5}$$

(iii) Let t be even. Let $\mathbf{A}^*$ be as in (ii). Let $w_{\rm od}, w_{\rm ev}$ be weights such that $w_{\rm od} \le w_h^*$ for odd $h = 1, 3, \dots, t-1$, and $w_{\rm ev} \le w_h^*$ for even $h = 2, 4, \dots, t-2$. By applying (i), construct a matrix $\mathbf{A}^{**}$ with $w_0^{**} = w_0^* = w_j$, $w_h^{**} = w_{\rm od}$ for odd h, $w_h^{**} = w_{\rm ev}$ for even $h \ge 2$. From $\mathbf{A}^{**}$ remove $t - 2f$ block rows and columns from the bottom and the right. A $2fd \times 2fd$ BDC matrix $\mathbf{A}'$ with $\overline{\mathbf{W}}(\mathbf{A}') = (w_j, w_{\rm od}, \underbrace{w_{\rm ev}, w_{\rm od}, \dots, w_{\rm ev}, w_{\rm od}}_{f-1 \text{ pairs}})$ is obtained. It is an incidence matrix of a configuration $v'_{k'}$ with

$$v' = 2fd,\ k' = w_j + w_{\rm od} + (f-1)(w_{\rm ev} + w_{\rm od}),\ f = 1, 2, \dots, t/2. \qquad (3.6)$$

Other methods for obtaining families of symmetric configurations from $\mathbf{A}$ of (3.2) can be found in [14, Sec. 4].

3.2 Using permutations of the set of integers $\{0, 1, \dots, v-1\}$

In this subsection we show a method to obtain BDC matrices from any (v, k) modular Golomb ruler with v a composite integer (see Theorem 3.5 below). A key tool is the notion of quotient modular Golomb ruler, as introduced in [37] and [48, p. 3]; it should be noted that quotient rulers are used in [37, 48] with a different goal, that is, in order to obtain difference triangle sets. We will construct a permutation σ of the set of indexes of points (and lines) of the cyclic configuration v_k associated to the original modular Golomb ruler, such that the incidence matrix $\mathbf{A}_\sigma$ of v_k corresponding to σ (cf. Definition 3.3) is a BDC matrix whose blocks correspond to the quotients of the original ruler.

We now sketch the construction of quotient rulers, as given in [37, 48]. For the ruler (3.1), and for any $h = 0, 1, \dots, t-1$, let

$$B_h = \left\{ \frac{a_i - h}{t} \mid a_i \equiv h \pmod t \right\},\ w_h = |B_h|. \qquad (3.7)$$

Clearly, $\sum_{h=0}^{t-1} w_h = k$ holds.

Theorem 3.2. [37, 48] *For every $h = 0, \dots, t-1$, B_h of (3.7) is a (d, w_h) modular Golomb ruler.*

Definition 3.3. Let $(a_1, a_2, \dots, a_k)$ be a (v, k) modular Golomb ruler. For each $u = 0, 1, \dots, v-1$, let

$$L_u = \{a_1 + u \pmod v,\ a_2 + u \pmod v, \dots, a_k + u \pmod v\}. \qquad (3.8)$$

For a permutation σ of the set $\{0, 1, \dots, v-1\}$, a $v \times v$ 01-matrix $\mathbf{A}_\sigma$ is defined as follows. Let $i, j \in \{0, 1, \dots, v-1\}$. The element in position (i, j) of $\mathbf{A}_\sigma$ is 1 if and only if $\sigma(j) \in L_{\sigma(i)}$ (or, equivalently, if and only if $\sigma(j) - \sigma(i) \pmod v \in L_0$).

Lemma 3.4. *For every choice of σ, the matrix $\mathbf{A}_\sigma$ of Definition 3.3 is a $\mathbf{J}_2$-free incidence matrix $\mathbf{M}(v, k)$ of a symmetric configuration v_k.*

Proof. By Theorem 1.5(ii), the matrix $\mathbf{A}_{id}$ is the incidence matrix of a cyclic symmetric configuration v_k. It is easily seen that $\mathbf{A}_\sigma$ is a different incidence matrix of the same v_k (points and lines are rearranged according to σ). $\square$

Theorem 3.5. *Let $(a_1, a_2, \ldots, a_k)$ be a (v, k) modular Golomb ruler with $v = td$. Let σ_t be the permutation of the set $\{0, 1, \ldots, v-1\}$ such that*

$$\sigma_t(ad + b) = bt + a \text{ for } 0 \le a \le t-1, \ 0 \le b \le d-1. \tag{3.9}$$

Let B_h and w_h be as in (3.7)*, and $\mathbf{A}_{\sigma_t}$ be as in Definition* 3.3*. Also, let $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_{t-1}$ be the $d \times d$ blocks of $\mathbf{A}_{\sigma_t}$ such that the first d rows of $\mathbf{A}_{\sigma_t}$ are a block row $[\mathbf{M}_0\mathbf{M}_1 \ldots \mathbf{M}_{t-1}]$. Finally, let $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{t-1}$ be the $d \times d$ blocks of $\mathbf{A}_{\sigma_t}$ such that the first d columns of $\mathbf{A}_{\sigma_t}$ are a block column $[\mathbf{M}_0\mathbf{T}_{t-1} \ldots \mathbf{T}_2\mathbf{T}_1]^{tr}$. Then*

(i) *Each matrix $\mathbf{M}_h$ is a* circulant *$d \times d$ 01-matrix of weight w_h. The first row of $\mathbf{M}_h$ corresponds to the (d, w_h) modular Golomb ruler B_h.*

(ii) *Each matrix $\mathbf{T}_h$ is a* circulant *$d \times d$ 01-matrix of weight w_h obtained from $\mathbf{M}_h$ by a cyclic shift of rows to the right by one position.*

(iii) *The matrix $\mathbf{A}_{\sigma_t}$ is a block double-circulant incidence matrix $\mathbf{M}(v, k)$ of a symmetric configuration v_k with the following structure:*

$$\mathbf{A}_{\sigma_t} = \begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_1 & \mathbf{M}_2 & \ldots & \mathbf{M}_{t-2} & \mathbf{M}_{t-1} \\ \mathbf{T}_{t-1} & \mathbf{M}_0 & \mathbf{M}_1 & \ldots & \mathbf{M}_{t-3} & \mathbf{M}_{t-2} \\ \mathbf{T}_{t-2} & \mathbf{T}_{t-1} & \mathbf{M}_0 & \ldots & \mathbf{M}_{t-4} & \mathbf{M}_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{T}_2 & \mathbf{T}_3 & \mathbf{T}_4 & \ldots & \mathbf{M}_0 & \mathbf{M}_1 \\ \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T3} & \ldots & \mathbf{T}_{t-1} & \mathbf{M}_0 \end{bmatrix}. \tag{3.10}$$

Proof. **(i)** Let $i, j \in \{0, 1, \ldots, d-1\}$, $h = 0, 1, \ldots, t-1$. The position (i, j) in $\mathbf{M}_h$ is the position $(i, hd + j)$ in $\mathbf{A}_{\sigma_t}$. The value 1 appears in this position if and only if $\sigma_t(hd + j) \in L_{\sigma t(i)}$. By (3.9) and (3.8), $\sigma_t(hd + j) = jt + h$, $\sigma_t(i) = it$, and $L_{\sigma t(i)} = \{a_1 + it \pmod v, \ldots, a_k + it \pmod v\}$. So, $\sigma_t(hd + j) \in L_{\sigma t(i)}$ if and only if

$$jt \in \{a_1 - h + it \pmod v, \ldots, a_k - h + it \pmod v\}. \tag{3.11}$$

If $a_u \not\equiv h \pmod t$ then $t \nmid (a_u - h + it)$ and $jt = a_u - h + it \pmod v$ cannot occur. Therefore, the condition (3.11) is equivalent to

$$j \in \left\{ \frac{a_u - h}{t} + i \pmod d | a_u \equiv h \pmod t \right\}, \tag{3.12}$$

which proves the assertion.

(ii) Let $i, j \in \{0, \ldots, d-1\}$, $h = 1, 2, \ldots, t-1$. The position (i, j) in T_h is the position $((t-h)d + i, j)$ in A_{σ_t}. The value 1 appears in this position if and only if $\sigma_t(j)$ belongs to $L_{\sigma_t((t-h)d+i)}$. Since $\sigma_t((t-h)d+i) = it + t - h$, we have

$$L_{\sigma_t((t-h)d+i)} = L_{it+t-h} = \{a_1 + it + t - h \pmod{v}, \ldots, a_k + it + t - h \pmod{v}\}.$$

Then $\sigma_t(j) = jt$ belongs to $L_{\sigma_t((t-h)d+i)}$ if and only if

$$jt \in \{a_1 + it + t - h \pmod{v}, \ldots, a_k + it + t - h \pmod{v}\}.$$

Arguing as in (i), we obtain that this condition is equivalent to

$$j \in \left\{ \frac{a_u - h}{t} + i + 1 \pmod{d} | a_u \equiv h \pmod{t} \right\}, \tag{3.13}$$

which proves the assertion.

(iii) We need to show that for every pair (i, j), $i, j = 0, 1, \ldots, v-d-1$, the value in position (i, j) in $\mathbf{A}_{\sigma_t}$ is equal to that in position $(i+d, j+d)$. The value in position (i, j) is equal to 1 if and only if $\sigma_t(j) \in L_{\sigma_t(i)}$. Write $i = i_1 d + i_2$, $j = j_1 d + j_2$, with $0 \le i_1, j_1 \le t-2$, $0 \le i_2, j_2 \le d-1$. Then $\sigma_t(j) = j_2 t + j_1$ and $\sigma_t(i) = i_2 t + i_1$, and hence the value in position (i, j) is 1 if and only if $j_2 t + j_1 - i_2 t - i_1 \pmod{v} \in L_0$. Note that $i + d = (i_1+1)d + i_2$ and $j + d = (j_1+1)d + j_2$. Then the value in position $(i+d, j+d)$ is 1 if and only if $j_2 t + (j_1+1) - i_2 t - (i_1+1) \pmod{v} \in L_0$. Since $(j_1+1) - (i_1+1) = j_1 - i_1 \pmod{v}$ holds, the assertion is proven.

□

Example 3.6. Let p be a prime. Let g be a primitive element of F_p. The following Ruzsa's sequence [44],[16, Sec. 5.4],[45, Th. 19.19] forms a $(p^2-p, p-1)$ modular Golomb ruler:

$$e_u = pu + (p-1)g^u \pmod{p^2-p},\ u = 1, 2, \ldots, p-1,\ v = p^2 - p. \tag{3.14}$$

(i) In [48, Tab. 5], a proper divisor of $p-1$ is taken as t to obtain new (d, w_h) modular Golomb rulers. In this case, $d = p\frac{p-1}{t}$ and $w_h = \frac{p-1}{t}$ for every h in (3.7). The matrix $\mathbf{A}_{\sigma_t}$ has a weight vector $\overline{\mathbf{W}}(\mathbf{A}_{\sigma_t}) = (\frac{p-1}{t}, \ldots, \frac{p-1}{t})$.

(ii) BDC matrices such that each weight w_h is in $\{0, 1\}$ admit an extension by Procedure E, see Section 2, and hence can be effectively used to obtain new families of configurations (cf. Section 4 and Example 4.4(ii)). There are two different possibilities to get a matrix $\mathbf{A}_{\sigma_t}$ with 01-weight vector from (3.14).

a) Fix $t = p-1$, $d = p$. Then for each $h = 0, 1, \ldots, t-1$ there is precisely one element e_u such that $e_u \equiv h \pmod{t}$. We have $e_{p-1} \equiv 0 \pmod{t}$ and $e_u \equiv u \pmod{t}$, $u = 1, 2, \ldots, p-2$. Therefore, $\overline{\mathbf{W}}(\mathbf{A}_{\sigma_{p-1}}) = (\underbrace{1, 1, \ldots, 1}_{p-1})$.

b) Fix $t = p$, $d = p-1$. In this case $e_u \not\equiv 0 \pmod{t}$ for all u. Also, for each $h = 1, 2, \ldots, t-1$ there is precisely one element e_u such that $e_u \equiv h \pmod{t}$. We have $e_u \equiv h \pmod{s}$ if and only if $-g^u \equiv h \pmod{p}$. Therefore, $\overline{\mathbf{W}}(\mathbf{A}_{\sigma_p}) = (0, \underbrace{1, 1, \ldots, 1}_{p-1})$.

Example 3.7. Consider the (q^2-1, q) modular Golomb ruler obtained from the cyclic starred affine plane $AG(2,q)$ [8]. Let $q^2-1 = td$. In [37], by using both counting arguments and properties of the ruler as a difference set, it is proved that if t is a divisor of $q+1$ then exactly $t-1$ values of w_h are equal to $\frac{q+1}{t}$, and there exists precisely one h_0 with $w_{h_0} = \frac{q+1}{t} - 1$. In [37] only proper divisor t of $q+1$ are considered, as this is the relevant case in connection with difference triangle sets. Yet, the same arguments work $t = q+1$, and hence on can obtain a weight vector of $\mathbf{A}_{\sigma_{q+1}}$ consisting of zeroes and units, and admitting an extension by Procedure E. Without loss of generality $\overline{\mathbf{W}}(\mathbf{A}_{\sigma_{q+1}}) = (0, \underbrace{1, 1, \ldots, 1}_{q})$ can be assumed. For comparison, see also Example 3.12 below.

3.3 Using subgroups of the automorphism group of a cyclic configuration

The geometrical interpretation of the procedure illustrated in Subsection 3.2 was presented in [12, 14]. Here, after summarizing some of the results from [12, 14], we apply the procedure to cyclic configurations $(q^2-1)_q$ associated to affine planes $AG(2,q)$ for t a divisor of $q-1$, see Theorem 3.11.

For a *cyclic* symmetric configuration v_k, viewed as an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$, let σ be a permutation of $\mathcal{P}$ mapping lines to lines, and acting regularly on both $\mathcal{P}$ and $\mathcal{L}$. Let S be the cyclic group generated by σ. Let $\mathcal{P} = \{P_0, \ldots, P_{v-1}\}$ and $\mathcal{L} = \{\ell_0, \ldots, \ell_{v-1}\}$. Arrange indexes so that $\sigma : P_i \mapsto P_{i+1 \pmod{v}}$ and $\ell_i = \sigma^i(\ell_0)$. Clearly, $P_i = \sigma^i(P_0)$ holds.

For any divisor d of v, the group S has a unique cyclic subgroup $\widehat{S}_d$ of order d, namely the group generated by σ^t where $t = v/d$. Let $O_0, O_1, \ldots, O_{t-1}$ (resp. $L_0, L_1, \ldots, L_{t-1}$) be the orbits of $\mathcal{P}$ (resp. $\mathcal{L}$) under the action of $\widehat{S}_d$. Clearly, $|O_i| = |L_i| = d$ for any i. We arrange indexes so that $P_0 \in O_0$, $O_w = \sigma^w(O_0)$, $\ell_0 \in L_0$, $L_w = \sigma^w(L_0)$. For each $i = 0, 1, \ldots, t-1$,

$$O_i = \{P_i, \sigma^t(P_i), \sigma^{2t}(P_i), \ldots, \sigma^{(d-1)t}(P_i)\},\ L_i = \{\ell_i, \sigma^t(\ell_i), \sigma^{2t}(\ell_i), \ldots, \sigma^{(d-1)t}(\ell_i)\}.$$

Equivalently, O_i (resp. L_i) consists of d points P_u (resp. d lines L_u) with u equal to i modulo t.

Let

$$w_u = |\ell_0 \cap O_u|,\ u = 0, 1, \ldots, t-1. \tag{3.15}$$

Clearly, $w_0 + w_2 + \ldots + w_{t-1} = k$.

Theorem 3.8. [14] *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ be a cyclic symmetric configuration v_k with $v = td$. Let d, t, $\hat{S}_d$, O_i, L_i be as above.*

(i) *For any i and j, every line of the orbit L_i meets the orbit O_j in the same number of points $w_{j-i \pmod t}$ where w_u is defined by* (3.15)*.*

(ii) *The incidence matrix of $\mathcal{I}$ is a block double-circulant matrix $\mathbf{A}$ of type* (3.2) *where $\mathbf{C}_{i,j}$ is a* circulant *$d \times d$ matrix of weight $w_{j-i \pmod t}$, with w_u as in* (3.15)*.*

In order to use Theorem 3.8 effectively one should find intersection numbers of orbits of the cyclic subgroup $\widehat{S}_d$. For cyclic projective and starred affine planes useful results on these numbers are given e.g. in [11, 14] and in the references therein.

Example 3.9. We consider the projective plane $PG(2, q)$ as a cyclic symmetric configuration $(q^2 + q + 1)_{q+1}$ [49],[14, Sec. 5],[16, Sec. 5.5],[45, Th. 19.15]. In this case the group S is a Singer group of $PG(2, q)$.

(i) Let $t = 3$, $t|(q^2+q+1)$, $p \equiv 2 \pmod 3$, and let $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. In [14, Prop. 4] the following is proved: $w_{i_0} = (q + 2\sqrt{q} + 1)/3$, $w_{i_1} = w_{i_2} = (q - \sqrt{q} + 1)/3$, if $q = p^{4m+2}$; $w_{i_0} = (q - 2\sqrt{q} + 1)/3$, $w_{i_1} = w_{i_2} = (q + \sqrt{q} + 1)/3$, if $q = p^{4m}$. Now we use Theorem 2.1 and (ii) of Subsection 3.1. By (3.5) with $c = 2$, we obtain families of configurations v_k with parameters

$$v_k \ : \ v = 2\frac{q^2+q+1}{3}, \ k = \frac{2q+\sqrt{q}+2}{3} - \delta, \ \delta \geq 0, \ q = p^{4m+2}, \ p \equiv 2 \pmod 3;$$
$$v_k \ : \ v = 2\frac{q^2+q+1}{3}, \ k = \frac{2q-\sqrt{q}+2}{3} - \delta, \ \delta \geq 0, \ q = p^{4m}, \ p \equiv 2 \pmod 3.$$

(ii) Let $q = p^{2m}$ be a square. Let t be a prime divisor of $q^2 + q + 1$. Then t divides either $q + \sqrt{q} + 1$ or $q - \sqrt{q} + 1$. Assume that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$. By [14, Prop. 6], in this case $w_0 = (q + 1 \pm (1 - t)\sqrt{q})/t$, $w_1 = w_2 = \ldots = w_{t-1} = (q + 1 \pm \sqrt{q})/t$. Now we use Theorem 2.1 and (ii) of Subsection 3.1. By (3.5), we obtain a family of configurations v_k with parameters

$$\begin{aligned} v_k \ &: \ v = c\frac{q^2+q+1}{t}, \ k = \frac{q+1\pm(1-t)\sqrt{q}}{t} + (c-1)\frac{q+1\pm\sqrt{q}}{t} - \delta, \quad (3.16)\\ c \ &= \ 1, 2, \ldots, t, \ \delta \geq 0, \ q = p^{2m}, \ t \text{ prime.} \end{aligned}$$

The hypothesis that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$ holds e.g. in the following cases: $q = 3^4$, $t = 7$; $q = 2^8$, $t = 13$; $q = 5^4$, $t = 7$; $q = 2^{12}$, $t = 19$; $q = 3^8$, $t = 7$; $q = 2^{16}$, $t = 13$; $q = 17^4$, $t = 7$; $p \equiv 2 \pmod t$, $t = 3$.

(iii) Let q be a square. Let $v \geq 1$, $v|(q - \sqrt{q} + 1)$, and $t = \frac{1}{v}(q - \sqrt{q} + 1)$. Then $d = v(q+\sqrt{q}+1)$ and, by [14, Prop. 7], we have $w_0 = \sqrt{q}+v$, $w_1 = w_2 = \ldots = w_{t-1} = v$. Now using (ii) of Subsection 3.1, for $v = 1$ we obtain a family of configurations with parameters (2.6). The orbits $O_0, O_1, \ldots, O_{t-1}$ are Baer subplanes. Moreover, the case $v = 1$ admits an extension by Procedure E, see Section 2; it can be effectively used for obtaining families of configurations, see Section 4 and Example 4.4(iii).

(iv) In Table 1, parameters of configurations v'_n with BDC incidence matrices are given. We use both (ii) and (iii) of Subsection 3.1. The starting weights w^*_i are obtained by computer forming orbits of subgroups $\widehat{S}_d$ of a Singer group of $PG(2,q)$. For $q=81$ we use (3.16). The values k',v' are calculated by (3.5),(3.6). Only cases with $v' < G(k')$ are included in the tables. Then the smallest value $k^{\#}$ for which $v' < G(k^{\#})$ is found. As a result, each row of the table provides configurations v'_n with $v' < G(n)$, $n = k^{\#}, k^{\#}+1,\dots,k'$, see (i) of Subsection 3.1 and (3.4).

INSERT Table 1 HERE

Remark 3.10. In [43, Prop. 3, Th. 9], parity check matrices of LDPC codes based on the Hermitian curve in $PG(2,q^2)$ and consisting of square cyclic submatrices are constructed by geometrical tools that can be considered as special cases of the more general approach of Theorem 3.8. The mentioned parity check matrices are incidence matrices of non-symmetric configurations. It is possible that by dismissing some units in the matrix, BDC configurations could be obtained. This problem is not considered here, nor in [43]. It is interesting to note that the matrix of [43, Prop. 3] uses points belonging to the Hermitian curve, whereas point set of the symmetric configuration in [14, Ex. 3], whose parameters are as in (2.12), coincides with the complement of the same curve.

Throughout the rest of the section, cyclic starred affine planes $(q^2-1)_Q$ are considered. In [14, 37] useful results for t a divisor of $q+1$ are obtained. Theorem 3.11 below extends our knowledge on this topic and gives new results for t dividing $q-1$. The proof is placed in Appendix; it uses orbits of cyclic subgroup.

Theorem 3.11. *Let q be an odd square. Consider the cyclic symmetric configuration $(q^2-q)_q$ associated to the starred affine plane of order q. Let t be a divisor of $\sqrt{q}-1$, and let $d=(q^2-1)/t$. Let $\mathbf{A}$ be an incidence BDC matrix of this configuration as in Theorem 3.8(ii). Let $w_0,w_1,\dots,w_{t-1}$ be weights of the circulant $d\times d$ blocks of $\mathbf{A}$.*

(i) *Let $t=\sqrt{q}+1$. Then $w_0=1$, $w_j=\sqrt{q}-1$ for j odd, $w_j=\sqrt{q}+1$ for j even, $j=1,2,\dots,\sqrt{q}$.*

(ii) *Let $t=\frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q}\equiv 1 \pmod 4$. Then $w_0=\sqrt{q}$, $w_1,w_2,\dots,w_{t-1}=2\sqrt{q}$.*

(iii) *Let $t=\frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q}\equiv 3 \pmod 4$. Then $w_0=\sqrt{q}+2$, $w_j=2\sqrt{q}-2$ for j odd, $w_j=2\sqrt{q}+2$ for j even, $j=1,2,\dots,\frac{1}{2}(\sqrt{q}+1)-1$.*

(iv) *Let $t=\frac{1}{4}(\sqrt{q}+1)$, $\sqrt{q}\equiv 3 \pmod 4$.*

- *If $\frac{1}{4}(\sqrt{q}+1)$ is odd, then $w_0=3\sqrt{q}$, $w_1,w_2,\dots,w_{t-1}=4\sqrt{q}$.*
- *If $\frac{1}{4}(\sqrt{q}+1)$ is even, then $w_0=3\sqrt{q}+4$, $w_j=2\sqrt{q}-4$ for j odd, $w_j=2\sqrt{q}+4$ for j even, $j=1,2,\dots,\frac{1}{4}(\sqrt{q}+1)-1$.*

Example 3.12. We consider the the cyclic symmetric configuration $(q^2-q)_q$ associated to the starred affine plane of order q. [8],[16, Sec. 5.6],[45, Th. 19.17], see also [14, Ex. 5, Sec. 6]. In this case the group S is and affine Singer group of $AG(2,q)$.

(i) Let t be a divisor of $q+1$. In [14, Prop. 10] it is proven that $w_0=(q+1)/t-1$, $w_1=w_2=\ldots=w_{t-1}=(q+1)/t$, cf. Example 3.7 which uses results of [37] obtained by a different approach. Putting $t=q+1$ we obtain $d=q-1$ and $w_0=0$, $w_1=w_2=\ldots=w_q=1$. Now using (ii) in Subsection 3.1 one can obtain a family of configurations with parameters (2.9). Moreover, the case $v=1$ admits an extension by Procedure E, see Section 2; it can be effectively used for obtaining families of configurations, see Section 4 and Example 4.4(i).

(ii) Let q be an odd square. Let $t=\sqrt{q}+1$, $d=(\sqrt{q}-1)(q+1)$. By Theorem 3.11(i) and (iii) of Subsection 3.1 one can obtain a family of configurations with parameters

$$v_k: v=2f(\sqrt{q}-1)(q+1),\ k=(2f-1)\sqrt{q},\ f=1,2,\ldots,\frac{\sqrt{q}+1}{2},\ q \text{ odd square.} \tag{3.17}$$

By using Theorem 3.11(ii),(iii),(iv), together with (iii) of Subsection 3.1, we obtain the same parameters as in (3.17). But the structure of an incidence matrix $\mathbf{M}(v,k)$ is different from that arising from Theorem 3.11(i).

(iii) In Table 2, parameters of configurations v'_n with BDC incidence matrices are given. We use (iii) of Subsection 3.1. The starting weights w_i^* are obtained by computer through the constructions of the orbits of subgroups $\widehat{S}_d$ of the affine Singer group. Notations is as in Table 1.

INSERT Table 2 HERE

4 Constructing configurations v_k admitting an extension

Establishing whether a configuration admits an extension or not is not an easy task in the general case. In this section we deal with configurations admitting incidence matrices with special type, and we show that may admit several extensions.

Definition 4.1. Let $v=td$, $t\geq k$, $d\geq k-1$, and let v_k be a symmetric configuration. Let $\mathbf{M}(v,k)$ be an incidence matrix of v_k, viewed as a $t\times t$ block matrix, every block being of type $d\times d$. We say that $\mathbf{M}(v,k)$ has *Structure E* if every $d\times d$ block is either a permutation matrix $\mathbf{P}_d$ or the zero $d\times d$ matrix $\mathbf{0}_d$.

Lemma 4.2. *Let $v=td, t\geq k, d\geq k-1$, and let v_k be a symmetric configuration. Assume that $\mathbf{M}(v,k)$ is an incidence matrix of v_k having Structure E.*

(i) *The matrix $\mathbf{M}(v,k)$ admits $\theta(t,d,k):=t\cdot\lfloor d/(k-1)\rfloor\geq t\geq k$ extensions.*

(ii) *Let* $\mathbf{M}(v+\theta(t,d,k),k)$ *be the matrix obtained from (i) by applying* $\theta(t,d,k)$ *extensions. Then* $\mathbf{M}(v+\theta(t,d,k),k)$ *above admits* $\theta_2(t,d,k) := \lfloor \theta(t,d,k)/(k-1) \rfloor \geq 1$ *extensions.*

Proof. **(i)** We need to provide $\theta(t,d,k)$ pairwise disjoint E-aggregates of $\mathbf{M}(v,k)$. For each block of type $\mathbf{P}_d$, one can easily define a set $\mathcal{E}$ of $\lfloor d/(k-1) \rfloor$ disjoint E-aggregates consisting of row and columns with non-trivial intersection with $\mathbf{P}_d$. Let $\mathbf{B}$ be the 01-matrix of type $t \times t$ such that each entry corresponds to a $d \times d$ block in $\mathbf{M}(v,k)$: an entry is 1 if the corresponding block is of type $\mathbf{P}_d$, 0 otherwise. Each row and each column of $\mathbf{B}$ has weight k. Therefore, it is possible to obtain a permutation matrix $\mathbf{P}_t$ by dismissing some units in $\mathbf{B}$. The sets $\mathcal{E}$ defined from blocks $\mathbf{P}_d$ corresponding to units of $\mathbf{P}_t$ are clearly disjoint, and their union gives $t \cdot \lfloor d/(k-1) \rfloor$ not intersecting E-aggregates.

(ii) Theorem 2.4(ii) can be applied $\theta_2(t,d,k)$ times.

□

Corollary 4.3. *Let* $v = td$, $t \geq k$, $d \geq k-1$, *and let* v_k *be a symmetric configuration. Assume that an incidence matrix* $\mathbf{A}$ *of* v_k *is a BDC matrix as in (3.2) with weight vector* $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, \ldots, w_{t-1})$.

(i) *If all the weights* w_u *belong to the set* $\{0,1\}$, *then* $\mathbf{A}$ *admits* $t+1$ *extensions.*

(ii) *If* $\overline{\mathbf{W}}(\mathbf{A}) = (0,1,1,\ldots,1)$, *then one can obtain a family of symmetric configurations* v_k *with parameters*

$$v_k : v = cd + \theta,\ \ k = c-1-\delta,\ \ c = 2,3,\ldots,t,\ \ \theta = 0,1,\ldots,c+1,\ \ \delta \geq 0. \tag{4.1}$$

(iii) *If* $\overline{\mathbf{W}}(\mathbf{A}) = (1,1,\ldots,1)$, *then one can obtain a family of symmetric configurations* v_k *with parameters*

$$v_k : v = cd + \theta,\ \ k = c-\delta,\ \ c = 2,3,\ldots,t,\ \ \theta = 0,1,\ldots,c+1,\ \ \delta \geq 0. \tag{4.2}$$

Proof. The matrix $\mathbf{A}$ has clearly Structure E. Then (i) follows from Lemma 4.2, together with Theorem 2.4(i). As to (ii) and (iii), we use (ii) of Subsection 3.1. □

In order to obtain a configuration having Structure E from a given one, sometimes the procedures described in Subsection 3.1 are useful; see Example 4.4(iii) below.

Example 4.4. (i) We consider the starred affine plane of order q as a cyclic configuration $(q^2-1)_q$, see Examples 3.7 and 3.12(i). Let $t = q+1$, $d = q-1$, $w_0 = 0$, $w_1 = w_2 = \ldots = w_q = 1$. By Corollary 4.3(ii) we obtain a family of symmetric configurations v_k with parameters

$$v_k : v = c(q-1)+\theta,\ \ k = c-1-\delta,\ \ c = 2,3,\ldots,q+1,\ \ \theta = 0,1,\ldots,c+1,\ \ \delta \geq 0. \tag{4.3}$$

(ii) We consider Ruzsa's configuration $(p^2-p)_{p-1}$, see Example 3.6(ii). Put $t = p-1$, $d = p$. Then $w_0 = w_1 = \ldots = w_{p-2} = 1$. By Corollary 4.3(iii) we obtain a family with

$$v_k : v = cp + \theta,\ k = c - \delta,\ c = 2, 3, \ldots, p-1,\ \theta = 0, 1, \ldots, c+1,\ \delta \geq 0,\ p \text{ prime.} \tag{4.4}$$

If $t = p$, $d = p-1$ then $w_0 = 0$, $w_1 = \ldots = w_{p-1} = 1$. We obtain a family with

$$v_k : v = c(p-1)+\theta,\ k = c-1-\delta,\ c = 2, 3, \ldots, p,\ \theta = 0, 1, \ldots, c+1,\ \delta \geq 0,\ p \text{ prime.} \tag{4.5}$$

(iii) Let q be a square. We consider $PG(2,q)$ as a cyclic configuration $(q^2+q+1)_{q+1}$, see Example 3.9(iii). Let $v = 1$, $t = q - \sqrt{q} + 1$, $d = q + \sqrt{q} + 1$, $w_0 = \sqrt{q} + 1$, $w_1 = w_2 = \ldots = w_{t-1} = 1$. By (i) of Subsection 3.1 we can put $w'_0 = 1$ and obtain the weight vector $(1, 1, \ldots, 1)$. Now, by Corollary 4.3(iii), we obtain a family with

$$\begin{aligned} v_k \ \ : \ \ & v = c(q + \sqrt{q} + 1) + \theta,\ k = c - \delta,\ c = 2, 3, \ldots, q - \sqrt{q} + 1, \\ & \theta = 0, 1, \ldots, c+1,\ \delta \geq 0,\ q \text{ square} \end{aligned} \tag{4.6}$$

5 The spectrum of parameters of cyclic symmetric configurations

In order to widen the ranges of parameter pairs $\{v, k\}$ for which a cyclic symmetric configuration v_k exists, we consider a number of procedures that allow to define a new modular Golomb ruler from a known one. Some methods have already been introduced in the paper, see Theorem 2.1.

Here we first recall a result from [45], which describes a method to construct different rulers with the same parameters.

Theorem 5.1. [45] *If $(a_1, a_2, \ldots, a_k)$ is a (v, k) modular Golomb ruler and m and b are integers with $\gcd(m, v) = 1$ then $(ma_1+b \pmod v, ma_2+b \pmod v, \ldots, ma_k+b \pmod v)$ is also a (v, k) modular Golomb ruler.*

It should be noted that a (v, k) modular Golomb ruler can be a $(v + \Delta, k)$ modular Golomb ruler for some integer Δ [24]. This property does not depend on parameters v and k only. This is why Theorem 5.1 can be useful for our purposes.

Example 5.2. We consider the $(31, 6)$ modular Golomb ruler

$$(a_1, \ldots, a_6) = (0, 1, 4, 10, 12, 17)$$

obtained from $PG(2, 5)$, see [46]. We can apply Theorem 5.1 for $m = 19$, $b = 0$. The $(31, 6)$ modular Golomb ruler $(ma_1 \pmod{31}, \ldots, ma_6 \pmod{31})$ is

$$(a'_1, \ldots, a'_6) = (0, 4, 11, 13, 14, 19).$$

Now we take $\Delta = 4$ and calculate the set of differences $\{a'_i - a'_j \pmod{35} | \, 1 \leq i, j \leq 6; i \neq j\}$, that is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\}$. As the all differences are distinct and nonzero, the starting $(31, 6)$ modular Golomb ruler is also a $(35, 6)$ modular Golomb ruler.

Example 5.3. We take the $(57, 8)$ modular Golomb ruler $(a_1, \ldots, a_8) = (0, 4, 5, 17, 19, 25, 28, 35)$ obtained from $PG(2, 7)$. Apply Theorem 2.1 for $\delta = 1$, and remove the integer 35. A $(57, 7)$ modular Golomb ruler $(a'_1, \ldots, a'_7) = (0, 4, 5, 17, 19, 25, 28)$ is obtained. Now we take $\Delta = -2$. Due to Definition 2.5 and Theorem 2.6, instead of differences we calculate the set of sums $\{a'_i + a'_j \pmod{55} | \, 1 \leq i \leq j \leq 7\}$, that is $\{0, 4, 5, 8, 9, 10, 17, 19, 21, 22, 23, 24, 25, 28, 29, 30, 32, 33, 34, 36, 38, 42, 44, 45, 47, 50, 53, 56\}$. As the all sums are distinct, the $(57, 7)$ modular Golomb ruler $(0, 4, 5, 17, 19, 25, 28)$ is also a $(55, 7)$ modular Golomb ruler.

For $k \leq 81$, we performed a computer search starting from the (v, k) modular Golomb rulers corresponding to (2.1)–(2.3). For projective and affine planes, we got a concrete description of the ruler from [46]. For Ruzsa's construction, we used (3.14) of Example 3.6. For every starting (v, k) modular ruler we first considered all possible m with $\gcd(m, v) = 1$, and applied Theorem 5.1 for $b = 0$ to get new rulers with the same parameters v and k. Then, we checked whether this ruler was also a $(v + \Delta, k)$ for some Δ.

We obtained improvements on the known results for $k \geq 16$. For the sake of completeness, we summarize the known results about the case $k \leq 15$ in Table 3. Table 4 lists known and new results about $16 \leq k \leq 41$, whereas Table 5 deals with the case $42 \leq k \leq 83$.

As to Table3, the values of $v_\delta(k)$ are taken from [23, Tab. IV],[31, Tab. 2], [47, Tab. 1a], [51]. The values of v for which cyclic symmetric configurations v_k exist (resp. do not exist) are written in normal (resp. in italic) font. Moreover, $\overline{v}$ means that no configuration v_k exists while $\overline{v^c}$ notes that no cyclic configuration v_k exists. Data from [18, 24, 26, 29, 33, 38, 47] are listed in the 4-th column of the table. We take into account that an entry of the form "t+" in the row "n" of [47, Tab. 1] means the existence of a cyclic symmetric configurations v_n with $v \geq t$. Also, we use the following *non-existence* results: $\overline{\mathit{32}_6}$ [24, Th. 4.8]; $\overline{\mathit{33}_6}$ [33]; $\overline{\mathit{34}^c_6}$, $\overline{\mathit{59}^c_8}$-$\overline{\mathit{62}^c_8}$ [38]; $\overline{\mathit{75}^c_9}$-$\overline{\mathit{79}^c_9}$, $\overline{\mathit{81}^c_9}$-$\overline{\mathit{84}^c_9}$ [18]. The following theorem of [26] is taken into account.

Theorem 5.4. [26, Th. 2.4] *There is no symmetric configuration $(k^2 - k + 2)_k$ if $5 \leq k \leq 10$ or if neither k or $k - 2$ is a square.*

The values of k for which the spectrum of parameters of cyclic symmetric configurations v_k is completely known are indicated by a dot "$\centerdot$"; the corresponding values of $E_c(k)$ are sharp and they are noted by the dot "$\centerdot$" too. Values of v obtained by our search lies within the range of the known parameters (see the 5-th column of the table).

INSERT Table 3 HERE

In Table 4, for $16 \leq k \leq 41$, $P(k) \leq v < G(k)$, data on the existence of cyclic symmetric configuration v_k are given. The known results from (2.1)–(2.3) are written in

normal font; the entries v_a, v_b, and v_c means, respectively, that the relations (2.1),(2.2), and (2.3) are used. The values of v obtained in this work are given in bold font; the entry **v-w** notes an interval of sizes from **v** to **w** without gaps. If an already known value lies within an interval **v-w** obtained in this work, then it is written immediately before the interval. Also, some data on the nonexistence (including those arising from Theorem 5.4) are written in italic font, in the form $\overline{v}$ or $\overline{v^c}$. For $k = 16$ the value $v_\delta(k) = 255$ [47] is taken into account. The nonexistence of some projective planes by Bruck-Ryser theorem is also indicated.

INSERT Table 4 HERE

In Table 5, for $42 \le k \le 83$, the upper bounds on the cyclic existence bound $E_c(k)$ obtained in this work are listed.

INSERT Table 5 HERE

6 The spectrum of parameters of symmetric (non-necessarily cyclic) configurations

The known results regarding to parameters of symmetric configurations can be found in [1]–[5],[7]–[9],[12]–[14],[18, 19, 21, 33, 44, 45, 47],[23]–[31],[38]–[40]; see also the references therein.

The known families of configurations v_k were described in Section 2. In Table 6, for $k \le 37$, $P(k) \le v < G(k)$, values of v for which a symmetric configuration v_k from one of the families of Section 2 exists are given. A subscript of an entry indicates that a specific (2.i) is used: more precisely v_a indicates that v is obtained from (2.1), and similarly $v_b \to$ (2.2), $v_c \to$ (2.3), $v_d \to$ (2.4), $v_e \to$ (2.5), $v_f \to$ (2.6), $v_g \to$ (2.7), $v_h \to$ (2.8), $v_i \to$ (2.9), $v_j \to$ (2.10), $v_k \to$ (2.11), $v_l \to$ (2.12), $v_m \to$ (2.13). An entry with more than one subscript means that the same value can be obtained from different constructions. An entry of type $v_{\text{subscript}_1,\text{subscript}_2,\dots} - v'_{\text{subscript}_1,\text{subscript}_2,\dots}$ indicates that a whole interval of values from v to v' can be obtained from the constructions corresponding to the subscripts.

To save space, in Table 6 if a value belongs to an interval obtained by the Extension Construction of (2.13), then it is listed only once, even if it can be obtained from different constructions as well.

INSERT Table 6 HERE

In Table 7, for $P(k) \le v < G(k)$, parameters of the symmetric configurations v_k from Sections 3 and 4 are listed. An entry of type $v_{\text{subscript}}$ indicates that either relations (3.i),(4.j) or Tables 1, 2 are used. More precisely $v_n \to$ (3.16), $v_p \to$ (3.17), $v_r \to$ (4.3), $v_s \to$ (4.4), $v_t \to$ (4.5), $v_u \to$ (4.6), $v_v \to$ Table 1, $v_w \to$ Table 2. For $k \le 37$, we listed all the results we got, whereas for $k =$ 38-41, 49, 56 we only give some illustrative examples.

INSERT Table 7 HERE

We note that a number of parameters are new: 322_{16}, 458_{19}, $459_{19}, 482_{20}, 574_{22}, 674_{24}$, 782_{26}, 1066_{27}-1072_{27}, 1104_{27}-1106_{27}, 1066_{28}-1072_{28}, 1104_{28}-1109_{28}, 1142_{28}-1146_{28}, 1104_{29}-1109_{29}, 1142_{29}-$1146_{29}, 1180_{29}$-1183_{29}, 1220_{29}, 1142_{30}-1146_{30}, 1180_{30}-1183_{30}, 1218_{30}-1220_{30}, 1180_{31}-1183_{31}, 1218_{31}-1220_{31}, 1256_{31}, 1257_{31}, 1218_{32}-1220_{32}, 1256_{32}-$1257_{32}, 1294_{32}$, 1256_{33}, 1257_{33}, 1294_{33}, 1294_{34}, 1430_{34}-1434_{34}, 1472_{35}-1475_{35}, 1514_{36}-1516_{36}, 1556_{37}, 1557_{37}; sometimes the gaps in an interval arising from (2.13) are filled.

The new cyclic configurations from Table 4, like 382_{17}-390_{17}, 401_{18}, 405_{18}-407_{18}, 410_{18}, 412_{18}, 413_{18}, also fill some gaps in the known range of parameters.

Parameters of the family of Example 3.9(i) are too big to be included in Table 7. For the same reason, parameters for (3.16) are only reported for $q = 3^4$.

Finally, in Table 8, for $k \leq 37$, $P(k) \leq v < G(k)$, we summarize the data from Tables 3,4,6, and 7. Also, we use the following known results on existence of sporadic symmetric configurations: 45_7 [5]; 82_9 [19, Tab. 1]; 135_{12}, see [26] with reference to Mathon's talk at the British Combinatorial Conference 1987; 34_6 [36]. The non-existence of configuration 112_{11} is proved in [34].

INSERT Table 8 HERE

In Table 8, the values of k for which the spectrum of parameters of symmetric configurations v_k is completely known are indicated by a dot "$\bullet$"; the corresponding values of $E(k)$ are exact and they are indidcated by a dot as well. The filling of the interval $P(k)$–$G(k)$ is expressed as a percentage in the last column. It is interesting to note that such a percentage is quite high, and that most gaps occur for v close to $k^2 - k + 1$.

Appendix: Proof of Theorem 3.11

Let ξ be a primitive element of F_{q^2}. Let $\omega = \xi^{\frac{q+1}{2}}$, and $\theta = \omega^{q+1}$. Identify a point $(x, y) \in AG(2, q)$ with the element $z = x + \omega y \in F_{q^2}$. As $\omega^{q-1} = -1$ it is straightforward to check that $z^{q+1} = x^2 + \theta y^2$.

We need to consider the orbits of $F^*_{q^2}$ under the action of the cyclic group generated by σ^t where $\sigma(\xi^i) = \xi^{i+1}$. The orbit O_j of the element ξ^j is $\{\xi^j, \xi^{j+t}, \xi^{j+2t}, \ldots, \xi^{j+(\frac{q^2-1}{t}-1)t}\}$. Let μ be a primitive element in F_q.

(i) Assume that $t = \sqrt{q} + 1$. For each $z \in O_j$ we have $z^{q+1} = \xi^{j(q+1)}(\xi^{(q+1)(\sqrt{q}+1)})^h$ for some h. Also, ξ^{q+1} is a primitive element of F_q and $(\xi^{(q+1)(\sqrt{q}+1)})^h \in F^*_{\sqrt{q}}$. This means for each $j = 0, \ldots, \sqrt{q}$, the orbit O_j consists precisely of the elements z such that $z^{q+1} \in \mu^j F^*_{\sqrt{q}}$. Therefore, the following lemma holds.

Lemma 6.1. *The orbit O_j in $AG(2, q)$ consists of the union of the $\sqrt{q} - 1$ conics with equation $x^2 + \theta y^2 = \mu^j \alpha$ where $\alpha \in F^*_{\sqrt{q}}$.*

The final step is to compute the sizes of the intersections $|O_j \cap \ell|$ where ℓ is any line of $AG(2, q)$ not passing through the origin. These sizes are the integers $w_0, \ldots, w_{\sqrt{q}}$. Choose the line $\ell : x = 1$.

Lemma 6.2. $w_0 = 1$.

Proof. We prove that $|\ell \cap O_0| = 1$. Note that O_0 consists of the conics $C_\alpha : x^2 + \theta y^2 = \alpha$. The line $x = 1$ meets the conic C_α in one point if $\alpha = 1$. If $\alpha \neq 1$ then the intersection is empty since θ is not a square in F_q (and α is a square since it is an element of $F_{\sqrt{q}}$). $\square$

Lemma 6.3. *Let N_j be the number of non-squares in the set $\mu^j F^*_{\sqrt{q}} - 1$. Then $w_j = 2N_j$.*

Proof. The conic $C_\alpha : x^2 + \theta y^2 = \mu^j \alpha$ meets the line $x = 1$ in 0 or 2 points. The latter case occurs precisely when $\mu^j \alpha - 1$ is not a square. $\square$

In order to compute the integers N_j, the following lemmas will be useful.

Lemma 6.4. *The collection of sets $H_\beta = \{\frac{\mu}{1-\mu\beta} F^*_{\sqrt{q}} - 1 \mid \beta \in F_{\sqrt{q}}\}$ coincides with $\{\mu^j F^*_{\sqrt{q}} - 1 \mid j = 1, \dots, \sqrt{q}\}$.*

Proof. We need only need to show that sets $\frac{\mu}{1-\mu\beta} F^*_{\sqrt{q}}$ are pairwise distinct. Assume on the contrary that $\frac{\mu}{1-\mu\beta} = \alpha \frac{\mu}{1-\mu\gamma}$ for $\beta, \gamma \in F_{\sqrt{q}}$, $\alpha \in F^*_{\sqrt{q}}$. Then $\alpha(1 - \mu\beta) = 1 - \mu\gamma$ that is $\mu(-\alpha\beta + \gamma) = 1 - \alpha$. If $\alpha\beta - \gamma = 0$ then $\alpha = 1$ and hence $\beta = \gamma$. If $\alpha\beta - \gamma \neq 0$ then $\mu \in F_{\sqrt{q}}$, which is a contradiction. $\square$

Lemma 6.5. *Fix $\beta \in F_{\sqrt{q}}$ and $j \in \{1, \dots, \sqrt{q}\}$. Assume that the set $\frac{\mu}{1-\mu\beta} F^*_{\sqrt{q}} - 1$ coincides with $\mu^j F^*_{\sqrt{q}} - 1$. Then $1 - \mu\beta$ is a square if and only if j is odd.*

Proof. Note that $\frac{\mu}{1-\mu\beta} F^*_{\sqrt{q}} - 1$ coincides with $\mu^j F^*_{\sqrt{q}} - 1$ if and only if $\mu^{j-1}(1 - \mu\beta) \in F_{\sqrt{q}}$. Then $\mu^{j-1}(1 - \mu\beta)$ is a square. Whence the assertion follows. $\square$

Let M_β be the number non-squares in the set $\frac{\mu}{1-\mu\beta} F^*_{\sqrt{q}} - 1$. By the previous lemma, the set of integers M_β coincides with the set of integers N_j. Let $A = M_0 = N_1$. Next we show that every M_β is related to A.

Lemma 6.6. *If $1 - \mu\beta$ is a square in F_q then $M_\beta = A$. If $1 - \mu\beta$ is not a square in F_q then $M_\beta = \sqrt{q} - A$.*

Proof. A is the number of non-squares in the set $H_0 = \{\mu\alpha - 1 \mid \alpha \in F^*_{\sqrt{q}}\}$. For each $\beta \in F_{\sqrt{q}}$, this set coincides with $\{\mu(\alpha + \beta) - 1 \mid \alpha \in F^*_{\sqrt{q}}, \alpha \neq -\beta\} \cup \{\mu\beta - 1\}$. But since $\mu(\alpha + \beta) - 1 = \mu\alpha + \mu\beta - 1 = (1 - \mu\beta)(\frac{\mu}{1-\mu\beta}\alpha - 1)$ we have that

$$H_0 = (1 - \mu\beta)\{\frac{\mu}{1 - \mu\beta}\alpha - 1 \mid \alpha \in F^*_{\sqrt{q}}, \alpha \neq -\beta\} \cup \{\mu\beta - 1\},$$

that is $H_0 = (1 - \mu\beta)H_\beta \setminus \{-1\} \cup \{\mu\beta - 1\}$. Two cases have to be distinguished.

a) $1 - \mu\beta$ is a square. Then either $\frac{1}{\mu\beta - 1}$ and $\mu\beta - 1$ are both squares or are both non-squares. It follows that the number of non-squares in H_0 equals the number of non-squares in H_β. Therefore, $M_\beta = A$.

b) $1 - \mu\beta$ is not a square. As -1 is a square, $\mu\beta - 1$ is not a square. The number of non-squares in H_0 equals the number of squares in H_β plus 1. Then $M_\beta = \sqrt{q} - A$. $\square$

As a corollary to Lemmas 6.5 and 6.6, the following result is obtained.

Lemma 6.7. *If j is odd then $N_j = A$. If j is even then $N_j = \sqrt{q} - A$.*

By the above lemma *only two possibilities occur for w_j, namely $2A$ and $2(\sqrt{q} - A)$.* Next we calculate A. Let u_1 be the number of $\beta's$ such that $1 - \mu\beta$ is a square in $GF(q)$. Then, from $w_0 + w_1 + \ldots + w_{\sqrt{q}} = q$, we obtain

$$q = 1 + 2u_1A + 2(\sqrt{q} - u_1)(\sqrt{q} - A) = 1 + 4u_1A + 2q - 2\sqrt{q}(u_1 + A).$$

Hence,

$$0 = q + 1 + 4u_1A - 2\sqrt{q}(u_1 + A) = (\sqrt{q} - 2u_1)(\sqrt{q} - 2A) + 1$$

Since both $\sqrt{q} - 2u_1$ and $\sqrt{q} - 2A$ are integers, the only possibility is that they are both equal to ± 1. This implies that $A = \frac{\sqrt{q}\pm 1}{2}$, $u_1 = \frac{\sqrt{q}\mp 1}{2}$ is the only solution. So, we have proved that the integers $w_0, \ldots, w_{\sqrt{q}}$ are such that: 1 occurs precisely once; the integer $\sqrt{q} + 1$ occurs $\frac{\sqrt{q}-1}{2}$ times; the integer $\sqrt{q} - 1$ occurs $\frac{\sqrt{q}+1}{2}$ times. Finally, since $w_j = w_{j'}$ if $j = j'$ (mod 2) and the number of odd integers in $[1, \sqrt{q}]$ is greater than that of even integers, the assertion of Theorem 3.11(i) follows.

(ii) Assume that $t = \frac{1}{2}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 1 \pmod 4$. An orbit here is the union of two orbits O_j of case (i). More precisely, an orbit consists of the union of the $2(\sqrt{q} - 1)$ conics with equation

$$x^2 + \theta y^2 = \mu^j \alpha, \qquad \alpha \in F^*_{\sqrt{q}} \cup \mu^{\frac{\sqrt{q}+1}{2}} F^*_{\sqrt{q}}.$$

Equivalently, an orbit here is the union $O_j \cup O_{j+\frac{\sqrt{q}+1}{2}}$ for some $j = 0, \ldots, \frac{\sqrt{q}-1}{2}$. Assume that $j > 0$. Since j and $j + \frac{\sqrt{q}+1}{2}$ are different modulo 2, we have that the number of points of ℓ_i in this orbit is $(\sqrt{q} - 1) + (\sqrt{q} + 1) = 2\sqrt{q}$. If $j = 0$, since $\frac{\sqrt{q}+1}{2}$ is odd we have that ℓ_1 meets $O_0 \cup O_{\frac{\sqrt{q}+1}{2}}$ in $\sqrt{q}$ points.

(iii) Assume that $t = \frac{1}{2}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 3 \pmod 4$. Again, an orbit here is the union $O_j \cup O_{j+\frac{\sqrt{q}+1}{2}}$ for some $j = 0, \ldots, \frac{\sqrt{q}-1}{2}$. Assume that $j > 0$. Since j and $j + \frac{\sqrt{q}+1}{2}$ are equal modulo 2, we have that the number of points of ℓ_i in this orbit is $2\sqrt{q} - 2$ if j is odd, $2\sqrt{q} + 2$ if j is even. If $j = 0$, since $\frac{\sqrt{q}+1}{2}$ is even we have that ℓ_1 meets $O_0 \cup O_{\frac{\sqrt{q}+1}{2}}$ in $\sqrt{q} + 2$ points.

(iv) Assume that $t = \frac{1}{4}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 3 \pmod 4$. An orbit here is the union $O_j \cup O_{j+\frac{\sqrt{q}+1}{4}} \cup O_{j+\frac{\sqrt{q}+1}{2}} \cup O_{j+3\frac{\sqrt{q}+1}{4}}$, for some $j = 0, \ldots, \frac{\sqrt{q}-3}{4}$. Then it is easy to deduce the assertion of Theorem 3.11(iv). □

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Table 1. Parametrs of configurations v'_n with BDC incidence matrices, $v' < G(n)$, $n = k^{\#}, k^{\#}+1, \dots, k'$, by (ii) of Subsection 3.1 from the cyclic projective plane $PG(2,q)$

q	t	d	w_i^*	c	k'	v'	$G(k')$	$k^{\#}$	$G(k^{\#})$
32	7	151	0, 5, 5, 6, 5, 6, 6	6	25	906	961	25	961
37	3	469	16, 9, 13	2	25	938	961	25	961
43	3	631	19, 13, 12	2	31	1262	1495	30	1361
49	3	817	21, 16, 13	2	34	1634	1877	33	1719
61	3	1261	25, 21, 16	2	41	2522	2611	40	2565
64	3	1387	27, 19, 19	2	46	2774	3407	42	2795
67	3	1519	28, 19, 21	2	47	3038	3609	44	3193
73	3	1801	28, 27, 19	2	47	3602	3609	47	3609
79	3	2107	31, 21, 28	2	52	4214	4541	51	4381
81	7	949	4, 13, 13, 13, 13, 13, 13	6	69	5694	8291	58	5703
81	7	949	4, 13, 13, 13, 13, 13, 13	5	56	4745	5451	54	4747
97	3	3169	39, 28, 31	2	67	6338	7639	62	6431
103	3	3571	39, 28, 37	2	67	7142	7639	65	7187
107	7	1651	24, 15, 15, 13, 15, 13, 13	6	89	9906	13557	75	9965
107	7	1651	24, 15, 15, 13, 15, 13, 13	5	76	8255	10179	69	8291
109	3	3997	43, 36, 31	2	74	7994	9507	69	8291
109	7	1713	8, 15, 15, 19, 15, 19, 19	6	83	10278	12041	77	10409
121	7	2109	21, 20, 13, 13, 21, 13, 21	6	86	12654	13075	85	12821
127	3	5419	49, 43, 36	2	85	10838	12821	80	11127
128	7	2359	24, 21, 21, 14, 21, 14, 14	6	94	14154	15769	91	15085
137	7	2701	24, 15, 15, 23, 15, 23, 23	6	99	16206	17081	96	16243
139	3	6487	52, 39, 49	2	91	12974	15085	86	13075
151	3	7651	57, 43, 52	2	100	15302	17663	93	15453
163	3	8911	63, 49, 52	2	112	17822	27043	102	18437
151	7	3279	32, 19, 19, 21, 19, 21, 21	6	127	19674	28921	105	19769

Table 2. Parametrs of configurations v'_n with BDC incidence matrices, $v' < G(n)$, $n = k^{\#}, k^{\#}+1, \ldots, k'$, by (ii) and (iii) of Subsection 3.1 from the cyclic affine plane $AG(2,q)$

q	t	d	w_i^*	c	f	k'	v'	$G(k')$	$k^{\#}$	$G(k^{\#})$
31	3	320	14, 9, 8	2		22	640	713	21	667
37	3	456	16, 9, 12	2		25	912	961	25	961
49	4	600	16, 12, 9, 12	3		34	1800	1877	34	1877
49	6	400	4, 9, 12, 8, 8, 8	5		36	2000	2011	36	2011
53	4	702	17, 12, 10, 14	3		37	2106	2199	37	2199
61	3	1240	25, 16, 20	2		41	2480	2611	39	2505
67	3	1496	26, 24, 17	2		43	2992	3015	43	3015
71	5	1008	8, 14, 15, 16, 18	4		50	4032	4189	50	4189
73	3	1776	30, 21, 22	2		51	3552	4381	47	3609
79	3	2080	32, 25, 22	2		54	4160	4747	50	4189
79	6	1040	8, 14, 13, 14, 18, 12	5		56	5200	5451	56	5451
79	6	1040	18, 12, 8, 14, 13, 14		2	50	4160	4189	50	4189
81	4	1640	25, 20, 16, 20	3		57	4920	5547	55	5197
81	4	1640	25, 20, 16, 20		1	45	3280	3375	45	3375
81	8	820	16, 8, 8, 8, 9, 12, 8, 12	7		64	5740	7055	59	5823
81	8	820	16, 8, 8, 8, 9, 12, 8, 12	6		56	4920	5451	55	5187
83	8	861	6, 10, 10, 10, 15, 11, 10, 11	7		66	6027	7515	60	6039
83	8	861	6, 10, 10, 10, 15, 11, 10, 11	6		56	5166	5451	55	5187
89	4	1980	26, 25, 18, 20	3		62	5940	6431	60	6039
89	8	990	17, 8, 10, 12, 8, 10, 10, 14	7		65	6930	7187	64	7055
97	3	3136	37, 34, 26	2		63	6272	6783	62	6431
97	4	2352	29, 26, 20, 22	3		69	7056	8291	65	7187
97	6	1586	21, 20, 14, 16, 14, 12	5		69	7930	8291	69	8291
101	4	2550	30, 25, 20, 26	3		70	7650	8435	68	7913
101	4	2550	30, 25, 20, 26		1	55	5100	5197	55	5197
103	3	3536	41, 32, 30	2		71	7072	8661	65	7187
103	6	1768	24, 18, 14, 17, 14, 16	5		80	8840	11127	72	8947
103	6	1768	24, 18, 14, 17, 14, 16	4		66	7072	7515	65	7187
103	6	1768	24, 18, 14, 17, 14, 16		2	70	7072	8435	65	7187
107	8	1431	17, 15, 10, 13, 10, 12, 17, 13	7		77	10017	10409	76	10179
107	8	1431	17, 15, 10, 13, 10, 12, 17, 13		3	73	8586	9027	71	8661
109	3	3960	42, 30, 37	2		72	7920	8947	69	8291
109	4	2970	32, 26, 22, 29	3		76	8910	10179	72	8947
109	9	1320	8, 14, 20, 10, 13, 10, 12, 10, 12	8		78	10560	10599	78	10599
113	4	3192	32, 32, 24, 25	3		80	9576	11127	75	9965
113	7	1824	10, 20, 14, 17, 22, 16, 14	6		80	10944	11127	80	11127
121	4	3660	36, 30, 25, 30	3		86	10980	13075	80	11127
121	4	3660	36, 30, 25, 30		1	66	7320	7515	66	7515
121	5	2928	16, 24, 28, 25, 28	4		88	11712	13491	83	12041
125	4	3906	37, 32, 26, 30	3		89	11718	13557	83	12041
125	8	1953	24, 16, 14, 15, 13, 16, 12, 15	7		96	13671	16243	90	13935
125	8	1953	24, 16, 14, 15, 13, 16, 12, 15		3	93	11718	15453	83	12041

Table 3. The existence and nonexistence of cyclic symmetric configurations v_k, $k \le 15$, $P(k) \le v_\delta(k) \le v < G(k)$

k	$P(k)$	$v_\delta(k)$	$v < G(k)$ in literature	$v < G(k)$ in this work	$E_c(k) \le$	$G(k)$
2.	3	3			3.	3
3.	7	7			7.	7
4.	13	13			13.	13
5.	21	21	$21, \overline{\mathit{22}}$		23.	23
6.	31	31	$31, \overline{\mathit{32}}, \overline{\mathit{33}}, \overline{\mathit{34}^c}$		35.	35
7.	43	48	48-50	$49, 50$	48.	51
8.	57	57	$57, \overline{\mathit{58}}, \overline{\mathit{59}^c\text{-}\mathit{62}^c}, 63\text{-}68$	$64, 67, 68$	63.	69
9.	73	73	$73, \overline{\mathit{74}}, \overline{\mathit{75}^c\text{-}\mathit{79}^c}, 80, \overline{\mathit{81}^c\text{-}\mathit{84}^c}, 85\text{-}88$	$86, 87$	85.	89
10	91	91	$91, \overline{\mathit{92}}, 107\text{-}110$	109	107	111
11	111	120	$120, 133, 135\text{-}144$	$137, 139, 142\text{-}144$	135	145
12	133	133	$133, \overline{\mathit{134}}, 156, 158, 159, 161\text{-}170$	$158, 162\text{-}165, 167, 169, 170$	161	171
13	157	168	$168, 183, 193\text{-}212$	$197, 201, 203\text{-}212$	193	213
14	183	183	$183, \overline{\mathit{184}}, 225\text{-}254$	$226, 227, 231, 233\text{-}254$	225	255
15	211	255	$255, 267\text{-}302$	$278, 282, 284, 286, 287, 290\text{-}302$	267	303

Table 4. Values of v for which a cyclic symmetric configuration v_k exists, $16 \le k \le 41$, $P(k) \le v < G(k)$

k	$P(k)$	$P(k) \le v < G(k)$	$E_c(k) \le$	$G(k)$
16	241	$\overline{241}^c$-$\overline{254}^c$, 255_b, 272_c, 273_a, 288_b, 307_a, **318**, **320-329**, **331-354**	331	355
17	273	273_a, $\overline{274}$, 288_b, 307_a, 342_c, **343**, **353**, 360_b, **357-363**, 381_a, **365-398**	365	399
18	307	307_a, 342_c, 360_b, 381_a, **401**, **403**, **405-407**, **410**, **412-418**, **420-432**	420	433
19	343	$\overline{344}$, 360_b, 381_a, **455**, **457**, **464**, **467**, **468**, **470-477**, **479**, **481-492**	481	493
20	381	381_a, $\overline{382}$, **503**, 506_c, **508**, **513**, **516**, **519**, **520**, **525**, 528_b, **527-530**, **532**, 553_a, **534-566**	534	567
21	421	$\overline{422}$, 506_c, 528_b, 553_a, **592**, **597**, **601**, **602**, **606-609**, **611**, 624_b, 651_a, **614-666**	614	667
22	463	$\overline{463}$, $\overline{464}$, 506_c, 528_b, 553_a, 624_b, **640**, **644**, **645**, 651_a, **649-712**	649	713
23	507	$\overline{507}$, $\overline{508}$, 528_b, 553_a, 624_b, 651_a, **683**, **696**, **698**, **699**, **702**, **707**, **709-711**, 728_b, **713-744**	713	745
24	553	553_a, $\overline{554}$, 624_b, 651_a, 728_b, **739**, 757_a, **759**, **761**, **763**, **765-770**, **772**, 812_c, 840_b, **775-850**	775	851
25	601	624_b, 651_a, 728_b, 757_a, 812_c, **837**, 840_b, **842-844**, **846-854**, **856-863**, 871_a, 930_c, 960_b, **865-960**	865	961
26	651	651_a, $\overline{652}$, 728_b, 757_a, 812_c, 840_b, 871_a, **900**, **905-907**, **910**, **912**, **913**, **916**, **917**, **919-921**, **924**, **925**, **929**, 930_c, **932-941**, 960_b, **943-984**	943	985
27	703	728_b, 757_a, 812_c, 840_b, 871_a, 930_c, 960_b, **971**, **975**, **977**, **978**, **987**, **991**, 993_a, **994**, **997**, **1000**, **1001**, **1003-1006**, **1008**, **1010-1015**, **1017**, **1019**, 1023_b, 1057_a, **1021-1106**	1021	1107
28	757	757_a, $\overline{758}$, 812_c, 840_b, 871_a, 930_c, 960_b, 993_a, 1023_b, **1045**, 1057_a, **1063**, **1067**, **1070**, **1074**, **1075**, **1077**, **1079-1082**, **1085-1170**	1085	1171
29	813	$\overline{814}$, 840_b, 871_a, 930_c, 960_b, 993_a, 1023_b, 1057_a, **1146**, **1151**, **1152**, **1155-1158**, **1162-1167**, **1169**, **1172**, **1173**, **1175**, **1177**, **1180-1185**, **1187-1246**	1187	1247
30	871	871_a, $\overline{872}$, 930_c, 960_b, 993_a, 1023_b, 1057_a, **1198**, **1199**, **1219**, **1220**, **1224**, **1229**, **1235**, **1236**, **1238**, **1240**, **1241**, **1243**, **1248**, **1249**, **1251**, **1253**, **1255**, **1258**, **1261**, **1264-1267**, **1269-1272**, 1332_c, **1274-1360**	1274	1361
31	931	$\overline{932}$, 960_b, 993_a, 1023_b, 1057_a, **1324**, **1325**, 1332_c, **1341**, **1344**, **1345**, **1346**, **1348**, **1349**, 1368_b, 1407_a, **1351-1494**	1351	1495
32	993	993_a, $\overline{994}$, 1023_b, 1057_a, 1332_c, 1368_b, **1383**, **1393**, **1401**, 1407_a, **1409**, **1411**, **1414**, **1421**, **1424**, **1428**, **1429**, **1430**, **1432**, **1434**, **1438-1441**, **1443-1445**, **1447-1457**, **1459-1568**	1459	1569
33	1057	1057_a, $\overline{1058}$, 1332_c, 1368_b, 1407_a, **1492**, **1506**, **1507**, **1518**, **1521**, **1529**, **1533**, **1535**, **1540**, **1542**, **1545**, **1547-1553**, **1555**, **1557-1559**, **1561-1563**, **1565-1567**, **1569-1578**, **1580-1591**, 1640_c, 1680_b, **1593-1718**	1593	1719

Table 4 (continue). Values of v for which a cyclic symmetric configuration v_k exists, $16 \le k \le 41$, $P(k) \le v < G(k)$

k	$P(k)$	$P(k) \le v < G(k)$	$E_c(k) \le$	$G(k)$
34	1123	$\overline{\mathit{1123}}$, $\overline{\mathit{1124}}$, 1332_c, 1368_b, 1407_a, 1640_c, 1680_b, **1699**, 1723_a, **1725**, **1735**, **1739**, **1742**, **1747**, **1748**, **1750**, **1752**, **1755-1757**, **1759**, **1761**, **1765-1770**, **1772**, **1773**, **1777**, **1779-1785**, 1806_c, 1848_b, **1787-1876**	1787	1877
35	1191	$\overline{\mathit{1192}}$, 1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, **1781**, **1783**, **1788**, **1801**, **1805**, 1806_c, **1807**, **1810**, **1812**, **1814**, **1817**, **1823**, **182**5, **1826**, **1829-1833**, **1835-1840**, 1848_b, **1842-1856**, 1893_a, **1859-1974**	1859	1975
36	1261	1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, **1855**, **1860**, **1886**, 1893_a, **1902**, **1905**, **1907**, **1908**, **1912**, **1915- 1922**, **1925-1930**, **1932**, **1937-1939**, **1941-1952**, **1954**, **1955**, **1957-1959**, **1961-2010**	1961	2011
37	1333	$\overline{\mathit{1334}}$, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, **1973**, **1986**, **1989**, **2001**, **2006-2008**, **2010**, **2017**, **2018**, **2023**, **2024**, **2028**, **2031**, **2033**, **2036-2039**, **2041-2043**, **2045**, **2046**, **2048**, **2053**, **2054-2057**, **2059**, **2061**, **2063**, **2065-2074**, **2076-2083**, 2162_c, **2085-2198**	2085	2199
38	1407	1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, **2059**, **2061**, **2073**, **2088**, **2089**, **2092**, **2094**, **2096**, **2097**, **2099**, **2100**, **2101**, **2103**, **2105**, **2106**, **2108**, **2110**, **2111**, **2114-2116**, **2118**, **2124**, **2126-2129**, **2136-2139**, **2142-2153**, **2155-2157**, **2159**, **2161**, 2162_c, **2163**, **2164**, **2166-2170**, **2172**, **2174-2178**, 2208_b, 2257_a, **2180-2292**	2180	2293
39	1483	$\overline{\mathit{1483}}$, $\overline{\mathit{1484}}$, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, **2265**, **2278**, **2281**, **2287**, **2293**, **2294**, **2297**, **2300**, **2302**, **2304**, **2315**, **2317**, **2323**, **2324**, **2326**, **2330**, **2338**, **2340**, **2341**, **2344-2346**, **2348-2350**, **2352-2354**, **2358**, **2361-2364**, **2366-2369**, **2372-2383**, **2385-2393**, 2400_b, **2395-2401**, 2451_a, **2403-2504**	2403	2505
40	1561	$\overline{\mathit{1562}}$, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, **2326**, **2345**, **2372**, **2374**, **2389**, **2393**, **2396**, 2400_b, **2401**, **2404**, **2411**, **2414**, **2416**, **2417**, **2418**, **2423**, **2424**, **2427**, **2431**, **2435**, **2436**, **2438**, **2440-2444**, 2451_a, **2449-2453**, **2455**, **2459-2461**, **2464-2467**, **2471**, **2474-2480**, **2482-2484**, **2486**, **2487**, **2489-2522**, **2524-2564**	2524	2565
41	1641	$\overline{\mathit{1642}}$, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, **2345**, 2400_b, **2449**, 2451_a, **2460**, **2479**, **2480**, **2491**, **2494**, **2496**, **2499**, **2508-2511**, **2513**, **2516**, **2518-2521**, **2524**, **2525**, **2528-2540**, **2542**, **2544**, **2546-2548**, **2550-2553**, **2555-2562**, **2564-2575**, **2577-2610**	2577	2611

Table 5. Upper bounds on the cyclic existence bound $E_c(k)$, $42 \leq k \leq 83$

k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$
42	2632	2795	53	4463	4695	64	6796	7055	75	9883	9965
43	2860	3015	54	4513	4747	65	6853	7187	76	10023	10179
44	2917	3193	55	5195	5197	66	7279	7515	77	10229	10409
45	3280	3375	56	5341	5451	67	7359	7639	78	10395	10599
46	3353	3407	57	5501	5547	68	7463	7913	79	10800	10817
47	3453	3609	58	5551	5703	69	8111	8291	80	10977	11127
48	3765	3775	59	5612	5823	70	8125	8435	81	11396	11435
49	3839	3917	60	5687	6039	71	8288	8661	82	11443	11629
50	3871	4189	61	5994	6269	72	8694	8947	83	11593	12041
51	4308	4381	62	6150	6431	73	8813	9027			
52	4359	4541	63	6611	6783	74	8965	9507			

Table 6. Values of v for which a symmetric configuration v_k (cyclic or non-cyclic) of families of Section 2 exists, $k \le 37$, $P(k) \le v < G(k)$

k	$P(k)$	$P(k) \le v < G(k)$	$G(k)$
8	57	$57_a, 63_{b,e}, 64_m$-68_m	69
9	73	$73_a, 78_{f,g}, 80_{b,e}, 81_m$-$88_m$	89
10	91	$91_a, 98_h, 110_{c,d,e,i,m}$	111
11	111	$120_{b,e}, 121_m$-$133_m, 143_m$-144_m	145
12	133	$133_a, 156_m$-170_m	171
13	157	$168_{b,e}, 169_m$-$183_m, 189_f, 208_m$-212_m	213
14	183	$183_a, 210_f, 224_m$-254_m	255
15	211	$231_f, 240_m$-302_m	303
16	241	$252_{f,g}, 255_{b,e}, 256_m$-$321_m, 323_m$-$354_m$	355
17	273	$273_a, 288_{b,e}, 289_m$-$307_m, 323_m$-$381_m, 391_m$-398_m	399
18	307	$307_a, 342_m$-$381_m, 403_f, 414_m$-432_m	433
19	343	$360_{b,e}, 361_m$-$381_m, 434_f, 437_m$-$457_m, 460_m$-492_m	493
20	381	$381_a, 460_m$-$481_m, 483_m$- 566_m	567
21	421	483_m-666_m	667
22	463	506_m-$573_m, 575_m$-712_m	713
23	507	$528_{b,e}, 529_m$-$553_m, 558_f, 575_m$-744_m	745
24	553	$553_a, 589_f, 600_m$-$673_m, 675_m$-850_m	851
25	601	$620_{f,g}, 624_{b,e}, 625_m$-$651_m, 675_m$-$960_m$	961
26	651	$651_a, 702_m$-$781_m, 783_m$-984_m	985
27	703	$728_{b,e}, 729_m$-$757_m, 783_m$-$1065_m, 1073_m$-1103_m	1107
28	757	$757_a, 812_m$-$1065_m, 1073_m$-$1103_m, 1110_m$-$1141_m, 1147_m$-1170_m	1171
29	813	$840_{b,e}, 841_m$-$871_m, 899_m$-$1057_m, 1073_m$-$1103_m, 1110_m$-$1141_m,$ 1147_m-$1179_m, 1184_m$-$1219_m, 1221_m$-1246_m	1247
30	871	$871_a, 930_m$-$1057_m, 1110_m$-$1141_m, 1147_m$-$1179_m, 1184_m$-$1217_m,$ 1221_m-1360_m	1361
31	931	$960_{b,e}, 961_m$-$1057_m, 1147_m$-$1179_m, 1184_m$-$1217_m, 1221_m$-$1255_m,$ 1258_m-1494_m	1495
32	993	$993_a, 1023_{b,e}, 1024_m$-$1057_m, 1184_m$-$1217_m, 1221_m$-$1255_m,$ 1258_m-$1293_m, 1295_m$-1568_m	1569
33	1057	$1057_a, 1221_m$-$1255_m, 1258_m$-$1293_m, 1295_m$-1718_m	1719
34	1123	1258_m-$1293_m, 1295_m$-$1429_m, 1435_m$-1876_m	1877
35	1191	1295_m-$1407_m, 1435_m$-$1471_m, 1476_m$-1974_m	1975
36	1261	1332_m-$1407_m, 1476_m$-$1513_m, 1517_m$-2010_m	2011
37	1333	$1368_{b,e}, 1369_m$-$1407_m, 1517_m$-$1555_m, 1558_m$-2198_m	2199

Table 7. Parameters of new symmetric configurations v_k (cyclic and non-cyclic) from Sections 3 and 4

k	$P(k)$	$P(k) \le v < G(k)$	$G(k)$
8	57	63_r-68_r	69
9	73	80_r-88_r	89
10	91	$110_{r,s,t}$	111
11	111	120_r-$133_r, 143_s, 144_{r,s,t}$	145
12	133	156_s-$169_s, 156_{r,t}$-$170_{r,t}$	171
13	157	168_r-$183_r, 210_r$-212_r	213
14	183	225_r-$254_r, 238_s$-$253_s, 240_t$-254_t	255
15	211	240_r-$302_r, 255_s$-$301_s, 256_t$-302_t	303
16	241	255_r-$354_r, 272_s$-$289_s, 272_t$-290_t,304_s-$321_s, 306_t$-$354_t, 323_s$-354_s	355
17	273	288_r-$307_r, 323_s$-$361_s, 324_t$-$362_t, 324_r$-$381_r, 391_s$-$398_s, 396_{r,t}$-$398_{r,t}$	399
18	307	342_s-$361_s, 342_t$-$362_t, 342_r$-$381_r, 414_s$-$432_s, 418_{r,t}$-$432_{r,t}$	433
19	343	360_r-$381_r, 437_s$-$457_s, 440_{r,t}$-$492_{r,t}, 460_s$-$481_s, 483_s$-492_s	493
20	381	460_s-$481_s, 462_{r,t}$-$566_{r,t}, 483_s$-529_s	567
21	421	483_s-$529_s, 484_t$-$530_t, 484_r$-$666_r, 609_s$-$631_s, 616_t$-$639_t, 638_s$-$666_s, 640_w,$ 644_t-$666_t, 651_u$-666_u	667
22	463	506_s-$529_s, 506_t$-$530_t, 506_r$-$712_r, 638_s$-$661_s, 640_w, 644_t$-$668_t, 667_s$-$712_s,$ 672_t-712_t	713
23	507	528_r-$553_r, 576_r$-$744_r, 667_s$-$691_s, 672_t$-$697_t, 696_s$-$744_s, 700_t$-744_t	745
24	553	600_r-$850_r, 696_s$-$721_s, 700_t$-$726_t, 725_s$-$850_s, 728_t$-850_t	851
25	601	$624_p, 624_r$-651_r,676_r-$960_r, 725_s$-$751_s, 728_t$-$960_t, 754_s$-$865_s, 868_s$-$897_s,$ 899_s-$960_s, 906_v, 912_w, 938_v$	961
26	651	702_r-$984_r, 754_s$-$781_s, 756_t$-$984_t, 783_s$-$865_s, 868_s$-$897_s, 899_s$-984_s	985
27	703	728_r-$757_r, 783_s$-$865_s, 784_t$-$962_t, 784_r$-$1074_r, 868_s$-$897_s, 899_s$-$961_s,$ 999_s-$1027_s, 1008_t$-$1037_t, 1036_s$-$1065_s, 1044_t$-$1074_t, 1073_s$-$1103_s,$ $1080_{r,t}$-$1106_{r,t}$	1107
28	757	812_s-$841_s, 812_t$-$842_t, 812_r$-$1074_r, 868_s$-$897_s, 899_s$-$961_s, 870_t$-$962_t,$ 1036_s-$1065_s, 1044_t$-$1074_t, 1073_s$-$1103_s, 1080_t$-$1111_t, 1080_r$-$1170_r,$ 1110_s-$1141_s, 1116_t$-$1148_t, 1147_s$-$1170_s, 1152_t$-1170_t	1171
29	813	840_r-$871_r, 899_s$-$961_s, 900_t$-$962_t, 900_r$-$1057_r, 1073_s$-$1103_s, 1080_{r,t}$-$1111_{r,t},$ 1110_s-$1141_s, 1116_{r,t}$-$1148_{r,t}, 1147_s$-$1179_s, 1152_{r,t}$-$1185_{r,t}, 1184_s$-$1219_s,$ $1188_{r,t}$-$1246_{r,t}, 1221_s$-1246_s	1247
30	871	930_s-$961_s, 930_t$-$962_t, 930_r$-$1057_r, 1110_s$-$1141_s, 1116_{r,t}$-$1148_{r,t},$ 1147_s-$1179_s, 1152_{r,t}$-$1185_{r,t}, 1184_s$-$1217_s, 1188_{r,t}$-$1222_{r,t}, 1221_s$-$1360_s,$ $1224_{r,t}$-$1360_{r,t}, 1262_v$	1361

Table 7 (continue)**.** Parameters of new symmetric configurations v_k (cyclic and non-cyclic) from Sections 3 and 4

k	$P(k)$	$P(k) \leq v < G(k)$	$G(k)$
31	931	960_r-$1057_r, 1147_s$-$1179_s, 1152_{r,t}$-$1185_{r,t}, 1184_s$-$1217_s, 1188_{r,t}$-$1222_{r,t}, 1221_s$-$1255_s, 1224_{r,t}$-$1494_{r,t}, 1258_s$-$1494_s, 1262_v$	1495
32	993	1023_r-$1057_r, 1184_s$-$1217_s, 1188_{r,t}$-$1222_{r,t}, 1221_s$-$1255_s, 1224_{r,t}$-$1568_{r,t}, 1258_s$-$1293_s, 1295_s$-1568_s	1569
33	1057	1221_s-$1255_s, 1224_t$-$1395_t, 1224_r$-$1718_r, 1258_s$-$1293_s, 1295_s$-$1387_s, 1394_s$-$1718_s, 1400_t$-$1718_t, 1634_v$	1719
34	1123	1258_s-$1293_s, 1260_t$-$1370_t, 1260_r$-$1436_r, 1295_s$-$1369_s, 1394_s$-$1429_s, 1400_t$-$1436_t, 1435_s$-$1876_s, 1440_{r,t}$-$1876_{r,t}, 1634_v, 1800_{p,w}$	1877
35	1191	1295_s-$1369_s, 1296_t$-$1370_t, 1296_r$-$1407_r, 1435_s$-$1471_s, 1440_{r,t}$-$1477_{r,t}, 1476_s$-$1974_s, 1480_{r,t}$-$1974_{r,t}, 1800_p$	1975
36	1261	1332_s-$1369_s, 1332_t$-$1370_t, 1332_r$-$1407_r, 1476_s$-$1513_s, 1480_{r,t}$-$1518_{r,t}, 1517_s$-$1873_s, 1520_{r,t}$-$2010_{r,t}, 1880_s$-$2010_s, 2000_w$	2011
37	1333	1368_r-$1407_r, 1517_s$-$1555_s, 1520_t$-$1881_t, 1520_r$-$2198_r, 1558_s$-$1717_s, 1886_t$-$1928_t, 1720_s$-$1873_s, 1880_s$-$1921_s, 1927_s$-$2065_s, 1932_t$-$2022_t, 2024_t$-$2068_t, 2067_s$-$2113_s, 2070_t$-$2198_t, 2106_w, 2109_u$-$2147_u, 2115_s$-$2198_s, 2166_u$-2198_u	2199
38	1407	1560_r-$2292_r, 2166_u$-$2205_u, 2223_u$-$2263_u, 2280_u$-2292_u	2293
39	1483	1600_r-$2504_r, 2223_u$-$2263_u, 2280_u$-$2321_u, 2337_u$-$2379_u, 2394_u$-$2437_u, 2400_p, 2451_u$-$2495_u, 2480_w$	2505
40	1561	1640_r-$1928_r, 1932_r$-$2564_r, 2280_u$-$2321_u, 2337_u$-$2379_u, 2394_u$-$2437_u, 2400_p, 2451_u$-$2495_u, 2480_w, 2508_u$-$2553_u, 2522_v$	2565
41	1641	1680_r-$1723_r, 1764_r$-$1893_r, 1932_r$-$1975_r, 1978_r$-$2492_r, 2400_p, 2480_w, 2494_r$-$2610_r, 2522_v$	2611
46	2071	$2400_p, 2774_v$	3407
49	2353	2400_p	3917
56	3081	$4745_{n,v}, 4920_w, 5200_w$	5451

Table 8. The current known parameters of symmetric configurations v_k (cyclic and non-cyclic), an integrated table

k	$P(k)$	$P(k) \le v < G(k)$	$E(k) \le$	$G(k)$	filling
3.	7	$\overline{7}$	7.	7	100%
4.	13	13	13.	13	100%
5.	21	21, $\overline{22}$	23.	23	100%
6.	31	31, $\overline{32}$, $\overline{33}$, 34	35.	35	100%
7	43	$\overline{43}$, $\overline{44}$, 45, 48-50	48	51	75%
8	57	57, $\overline{58}$, 63-68	63	69	67%
9	73	73, $\overline{74}$, 78, 80-88	80	89	75%
10	91	91, $\overline{92}$, 98, 107-110	107	111	35%
11	111	$\overline{111}$, $\overline{112}$, 120-133, 135-144	135	145	76%
12	133	133, $\overline{134}$, 135, 156-170	156	171	47%
13	157	$\overline{158}$, 168-183, 189, 193-212	193	213	68%
14	183	183, $\overline{184}$, 210, 224-254	224	255	47%
15	211	$\overline{211}$, $\overline{212}$, 231, 240-302	240	303	72%
16	241	252, 255-354	255	355	89%
17	273	273, $\overline{274}$, 288-307, 323-398	323	399	78%
18	307	307, 342-381, 401, 403, 405-407, 410, 412-432	412	433	54%
19	343	$\overline{344}$, 360-381, 434, 437-492	437	493	53%
20	381	381, $\overline{382}$, 460-566	460	567	59%
21	421	$\overline{422}$, 483-666	483	667	75%
22	463	$\overline{463}$, $\overline{464}$, 506-712	506	713	84%
23	507	$\overline{507}$, $\overline{508}$, 528-553, 558, 575-744	575	745	84%
24	553	553, $\overline{554}$, 589, 600-850	600	851	85%
25	601	620, 624-651, 675-960	675	961	88%
26	651	651, $\overline{652}$, 702-984	702	985	85%
27	703	728-757, 783-1106	783	1107	88%
28	757	757, $\overline{758}$, 812-1170	812	1171	87%
29	813	$\overline{814}$, 840-871, 899-1057, 1073-1246	1073	1247	84%
30	871	871, $\overline{872}$, 930-1057, 1110-1360	1110	1361	78%
31	931	$\overline{931}$, $\overline{932}$, 960-1057, 1147-1494	1147	1495	68%
32	993	993, $\overline{994}$, 1023-1057, 1184-1568	1184	1569	73%
33	1057	1057, $\overline{1058}$, 1221-1718	1221	1719	76%
34	1123	$\overline{1123}$, $\overline{1124}$, 1258-1876	1258	1877	82%
35	1191	$\overline{1192}$, 1295-1407, 1435-1974	1435	1975	83%
36	1261	1332-1407, 1476-2010	1476	2011	81%
37	1333	$\overline{1334}$, 1368-1407, 1517-2198	1517	2199	84%